Canadian Studies in Population, Vol. 1, 1974, pp. 29-35.

A GENERAL CONDITION FOR STABILITY IN DEMOGRAPHIC PROCESSES*

Nathan Keyfitz
Harvard University, Cambridge, Massachusetts, U.S.A.

Résumé—On utilise la théorie graphique pour examiner en détail la propriété de la stabilité des déplacement démographiques.

Abstract—Graph theory is employed to investigate the stability property of population transfer.

Population analysis involves transfers, transitions, or transformations of individuals among categories or states. The common population projection is transfer among age groups (including as a transfer the production of children by adults in the reproductive ages); migration is transfer among countries or among subdivisions of a country; school population is transferred among types of school, among grades within schools, and ultimately is transformed into graduates with various formal qualifications; in mortality analysis the living population is transformed into the dead via the several statistically recognized causes; labour force analysts study transfers among the school and working populations, and within the working population among the several occupations.

In any of these or other cases, if the transfers take place at fixed rates they can be represented in three equivalent ways: (a) as a set of linear equations, (b) as a matrix, and (c) as a graph. Each of the three ways has its usefulness. For a population in which age only is recognized and the transfers are those of birth and death, the set of linear equations is the ordinary population projection. Less convenient for numerical work than linear equations because it includes many multiplications by zero, the matrix and its powers show the effect on later times and later generations of an individual initially at a given age. The graph is poor for computing, and in the usual simplification replaces all positive elements of the matrix by unity, but it permits an easy investigation of the stability property. It is this last that is our subject here.

The stability property is of interest because it tells us the result of indefinite repetition of a given set of transfers, and helps thereby the study of the present condition. If the net movement to British Columbia during a decade is (say) 0.02 of the population of the remainder of Canada, and this along with natural increase is the whole model, then stability will not occur except in the trivial sense that the whole of the remainder of Canada will be emptied into British Columbia. If the model is slightly complicated by the information that the net of 0.02 is the resultant of 0.025 of the population of the rest of Canada moving into British Columbia and 0.05 of the population of British Columbia moving out, then the projection is altogether changed, and continuance of the process will result in a steady state with a calculable fixed ratio of population between British Columbia and the rest of Canada.

Such a simple case will be convenient for purposes of exposition, and the results will be transferable to models recognizing age, region, rural and urban, school attendance and grade, labour force and occupation, or any combination of these and other characteristics. Two theorems will be presented, easy to apply though difficult to prove. Proofs will be beyond the scope of this paper, as on the farther side of the division of labour between demographers and mathematicians.

*An invited paper.

A Two-Subgroup Model

Within a population in which there exist statistically recognizable subpopulations—regions of a country, social classes, educational levels—we consider not changes within, but transfers among, such groupings. Given the rate of growth in each subpopulation and the rates of transfer among subpopulations, changes can be described by a set of differential or difference equations. For a simple special case, suppose two subpopulations p and q, where p_t and q_t denote the sizes of the two subpopulations at time t. If growth of the subpopulations is accompanied by migration in both directions then change in the system can be described by

$$\begin{aligned} p_{t+1} &= r_{11}p_t + r_{12}q_t, \\ q_{t+1} &= r_{21}p_t + r_{22}q_t. \end{aligned} \tag{1}$$

In general, $r_{11} > 0$ is the net rate of growth of the first subpopulation, i.e., the rate of increase after outward migration has been accounted for; $r_{12} > 0$ is the fraction of the second population transferring into the first, and similarly for r_{22} and r_{21}.

Equation (1) is identical to the matrix equation

$$\begin{bmatrix} p_{t+1} \\ q_{t+1} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} p_t \\ q_t \end{bmatrix}, \tag{2}$$

which can be compactly written

$$\underset{\sim}{P}_{t+1} = \underset{\sim}{R}\underset{\sim}{P}_t, \tag{3}$$

where $\underset{\sim}{P}_t$ is the population vector at time t, $\underset{\sim}{R}$ is the matrix of growth and transfer rates—referred to indifferently as a transition or a projection matrix—and $\underset{\sim}{P}_{t+1}$ is the population vector at time $t + 1$. If the rates r_{11}, r_{21}, r_{12}, and r_{22} are fixed over time, then (3) recurrently determines the population at any arbitrary time subsequent to t. The population at time $t + n$ equals that at time t successively operated on by $\underset{\sim}{R}$:

$$\underset{\sim}{P}_{t+n} = \underset{\sim}{R}(\ldots(\underset{\sim}{R}(\underset{\sim}{R}\underset{\sim}{P}_t))\ldots) = \underset{\sim}{R}^n\underset{\sim}{P}_t. \tag{4}$$

If $r_{12} = r_{21} = 0$, and $r_{11} \neq r_{22}$, then it is readily shown that the two subpopulations never come into a fixed ratio to one another, but the subgroup with the larger rate keeps growing relative to the one with the lower rate. But if r_{12} and r_{21} are positive, then no matter how different r_{11} and r_{22} may be, the two subpopulations will ultimately come to increase at the same rate. This stability of the ratio of the one population to the other will of course occur more quickly if r_{12} and r_{21} are large in relation to the difference between r_{11} and r_{22}.

More Complex Cases and the Use of Graphs

In the preceding example we have been dealing with two subpopulations, a case simple enough that we could demonstrate stability or the lack of it by easy matrix multiplication. Problems arise involving scores, even thousands, of subgroups within a population; the matrices become very large and the conditions for stability too complex for the problem to be tackled as was done above. Fortunately, two general rules suffice to determine which projection matrices will result in stability, rules expressed in terms of graphs.

A graph consists of *vertices,* corresponding to the states or subgroups of a population, and *edges,* representing transitions among states, regions, etc. Formally the vertices and edges are not defined except as points and the lines joining them. The edges will be *directed,* the direction indicated by an arrow (Malkevitch and Meyer, 1974; Berge, 1963).

An exact correspondence can be drawn between a graph and a matrix containing zeros and ones. Consider the 3×3 matrix $\underset{\sim}{A}$ and an initial population vector $\underset{\sim}{P}$,

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } \underset{\sim}{P} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

The positive entry equal to one in position a_{13} of $\underset{\sim}{A}$ enters the matrix multiplication $\underset{\sim}{A}\underset{\sim}{P}$ by multiplying p_3, and the result is included in the first element of $\underset{\sim}{A}\underset{\sim}{P}$. That is, a positive entry in a_{13} represents a transition from state 3 in the initial vector to state 1 in the final vector. The matrix $\underset{\sim}{A}$ provides for transitions from state 1 to itself, from 3 to 1, from 1 to 2, and from 2 to 3. In general, a positive entry in position a_{ij} (row i, column j) represents a transition from state j to state i; if $i = j$, it represents a transition from state i to itself. On the other hand, a zero entry in a_{ij} means that no transition is permitted from state j to state i. Denoting states by vertices and possible transitions by arrows, we see that the matrix $\underset{\sim}{A}$ is equivalent to the graph of Fig. 1.

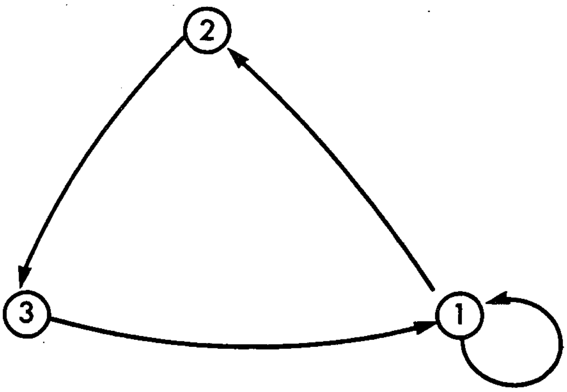

FIGURE 1. GRAPH CORRESPONDING TO MATRIX $\underset{\sim}{A}$

Irreducibility or Connectivity

A necessary condition for ultimate stability is that the projection matrix and corresponding graph be *irreducible*. A matrix and its corresponding graph are irreducible if passage is possible from any vertex to any other vertex, with or without going through further vertices on the way; a graph not satisfying this condition is *reducible*. Thus the graph in Fig. 1 is irreducible. A graph is reducible, for instance, if it divides into separate noncommunicating blocks as in Fig. 2a, or if it has even one point out of which no passage is provided, as vertex D in Fig. 2b. Anyone who prefers can replace the word "irreducible" by "connected," the former being more often used in reference to matrices, the latter in reference to graphs.

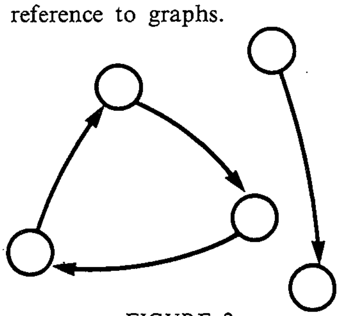

FIGURE 2a

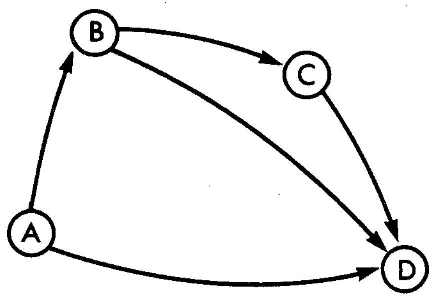

FIGURE 2b

REDUCIBLE GRAPHS

Irreducibility of a matrix is a necessary but not sufficient condition for stability to occur at high powers of the matrix. What is further required is that the matrix fill up with positive elements, so that it contains no zeros as it goes to high powers. Consider the matrix

$$\underset{\sim}{C} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and the corresponding irreducible graph in Fig. 3:

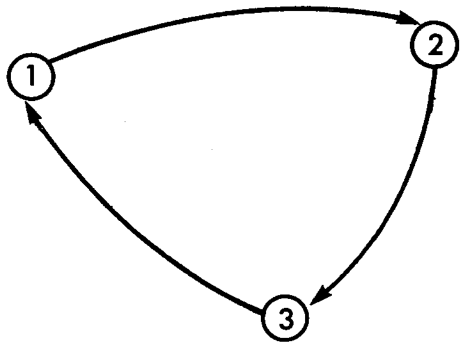

FIGURE 3. A CYCLIC GRAPH C

$\underset{\sim}{C}^3$ is the identity matrix, and thus $\underset{\sim}{C}^4$ is the same as $\underset{\sim}{C}$. Though each cell becomes unity at one time or another, the matrix at no time fills up with positive numbers; instead, it goes through endless cylces. Repeated multiplication of a population vector by $\underset{\sim}{C}$ will never produce stable ratios.

Primitivity

By permitting one element of $\underset{\sim}{C}$ to communicate with itself we destroy the cyclic character of $\underset{\sim}{C}$. For example,

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is not cyclic. In particular, $\underset{\sim}{A}^8$ contains only positive numbers, and this special type of irreducible matrix is called *primitive*. In terms of an arbitrary matrix $\underset{\sim}{A}$, primitivity means that some power of $\underset{\sim}{A}$, say $\underset{\sim}{A}^N$, can be found that has only positive elements. For the corresponding graph primitivity implies that a number N can be found such that one can go from any vertex of the graph to any other vertex in exactly N moves.

Irreducibility requires only that every point be reachable from every other point, not necessarily in the same number of moves; in the irreducible graph C of Fig. 3 one can go from 1 to 2 in one move but not in two, and from 2 to 1 in two moves but not in one. Primitivity requires the existence of a number N such that a route of *length* N on the graph can be found between any two points, a special case of irreducibility. We will accept without proof the fact that primitivity of a non-negative matrix such as a projection matrix is a sufficient condition for stability (Perron, 1907; Frobenius, 1912; Parlett, 1970).

Like irreducibility primitivity may be recognized from a graph. A *circuit* of a graph is a unidirectional closed subset of vertices and directed edges whose length is equal to the number of edges of the subset. Thus, if vertex A connects with itself it forms a circuit of length one, whereas the circuit in Fig. 3 is of length three. A fundamental theorem expresses primitivity in terms of the lengths of circuits in a graph: an irreducible graph, or its matrix, is primitive *if and only if it has at least two circuits whose lengths* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are relatively prime*. (Two numbers are relatively prime if they have no common divisor greater than one.)

This theorem has essentially the same content as a proposition in number theory: Given integers $\mathcal{L}_1$ and $\mathcal{L}_2$ that are relatively prime, positive integers a and b can be found to make the weighted sum $a\mathcal{L}_1 + b\mathcal{L}_2$ equal to any arbitrarily chosen integer greater than some integer I. This proposition from number theory implies that the existence of two circuits of relatively prime lengths $\mathcal{L}_1$ and $\mathcal{L}_2$ guarantees that the graph will be primitive, and that the corresponding matrix will fill up with positive numbers at and beyond some power N.

Application to Birth and Death

Consider the above theorem in terms of the survival and aging process. Aging corresponds to a graph that goes from the first to the second, to the third, to the fourth, . . . , age groups (Fig. 4). This graph is neither irreducible nor primitive; everyone

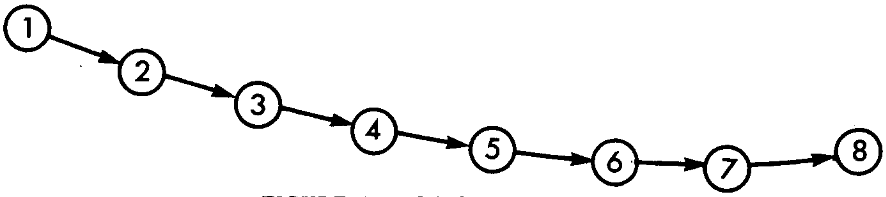

FIGURE 4. AGING PROCESS

drifts to the end of life. Suppose a provision for birth in the sixth age group, and hence one circuit (Fig. 5). Since the seventh and eighth age groups lead nowhere we drop these ages

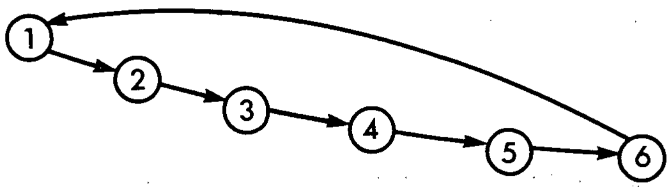

FIGURE 5. AGING WITH ONE AGE OF CHILDBEARING

and the graph up to age 6 becomes irreducible. However, since it has only one circuit, the graph is not yet primitive. In order to obtain primitivity we need at least two age groups of childbearing, and we must be careful to choose these age groups relatively prime, e.g., the third and sixth ages would not suffice. Allowing for fertility in the fifth and sixth ages the graph becomes that shown in Figure 6 and now the graph and the corresponding matrix are primitive.

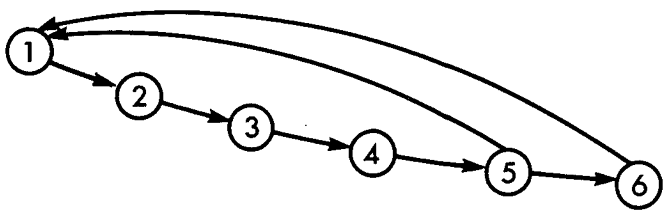

FIGURE 6. AGING WITH TWO AGES OF CHILDBEARING

To sum up, an irreducible matrix that is also primitive makes any vector that it is premultiplying come to a fixed set of ratios, i.e., a stable population, and one that does not depend on the original composition of the population. This ergodic property, the tendency of a vector or population to forget its past, is easily illustrated by examples in which the same matrix was applied repeatedly. An analogous property of a series of different matrices was conjectured by Coale: that two different vectors become similar to one another when they are acted on by the same series of matrices. This important theorem was proven by Lopez (1961). Note that the considerations of this article prove ergodicity with changing matrices, provided the same elements are always non-zero. Lopez's proof avoids this and other restrictions.

The results of the present section apply wherever we seek to understand a current movement by asking what would happen if it continued indefinitely into the future. Any linear projection with fixed transitions, recognizing ages, regions, occupations, school grades, etc., or any combination of these, may be treated in the same way. Using the methods of this section, we might study, for example, migration between California and the rest of the United States over a decade, as did Rogers (1968), and ask what would be the stable condition if both in-migration and out-migration continued indefinitely; we might study movement through the educational system (Stone, 1966); through the occupation structure (Prais, 1955; Tabah, 1968); and many others. In all such cases the transition process can be represented as a matrix, and the stability properties of the matrix studied by means of a graph.

Strong ergodicity is the approach to stability with the repeated action on a given vector of a given (primitive) matrix. Of more practical application is weak ergodicity, the result of the action of a given sequence of different matrices. When two different initial age distributions, say P_0 and P_0', are both acted on by the sequence of matrices M_0, $M_1, \ldots M_n$, then weak ergodicity means that they come to resemble one another. In symbols, if n is large enough, the equation

$$M_n \ldots M_2 M_1 M_0 P_0 = M_n \ldots M_2 M_1 M_0 P_0'$$

represents weak ergodicity. The past is forgotten in the sense that the difference between P_0 and P_0' disappears in the products, provided n is large. The vectors resulting from the multiplication are purely a function of the sequence of matrices, and not of the initial vector. This is a major generalization of strong ergodicity, which may be represented as $M^n P_0 = M^n P_0'$, n large.

David McFarland suggests that weak ergodicity can be proved in terms of strong. He would start by defining the *intersection* of two or more graphs as that graph which has only those points and lines common to all of the graphs being intersected. The points are the ages that are recognized, and we suppose that all matrices of the sequence are of the same size and refer to the same age groups. The lines are determined by the non-zero elements in the matrix; suppose that all of the matrices have positive fertility in the fourth and fifth age groups, or some other pair of ages prime to one another as long as it is the same for all the matrices. Other non-zero elements, either the same from matrix to matrix or differing, cannot interfere with ergodicity, so the existence of two common relatively prime ages of positive fertility is a sufficient condition. More generally, the primitivity of the intersection matrix of the sequence M_0, $M_1, \ldots M_n$ is sufficient for weak ergodicity.

Strong ergodicity has the convenience that the proportional distribution of the resultant product vector, $M^n P_0$, is a simple function of the elements of M as n becomes large. Where a one-sex age distribution is under discussion, this is the stable age distribution of much importance in demography. In the same way the product vector $M_n \ldots M_1 M_0 P_0$ is purely a function of the Ms, and not of P_0, but no simple expression in terms of the elements of the Ms can be hoped for.

Acknowledgments

I am grateful to Walter Meyer and Noreen Goldman for suggestions on this paper.

References

Berge, Claude. 1963. Théorie des graphes et ses applications. 2nd edition, Paris: Dunod.

Coale, Ansley J. 1957. How the Age Distribution of Human Population is Determined. Cold Spring Harbor Symposia on Quantitative Biology 22: 83-89.

Frobenius, G. 1912. Über Matrizen aus nicht negativen Elementen, Sitz. Preuss. Akad. 456-477.

Lopez, Alvaro. 1961. Problems in Stable Population Theory. Princeton: Office of Population Research.

Malkevitch, Joseph and Walter Meyer. 1974. Graphs, Models, and Finite Mathematics. Englewood Cliffs, New Jersey: Prentice-Hall.

McFarland, D. D. 1969. On the Theory of Stable Populations. A New and Elementary Proof of the Theorems under Weaker Assumptions, Demography 6: 303-322.

Parlett, Beresford. 1970. Ergodic Properties of Populations: The One Sex model. Theoretical Population Biology 1: 191-207.

Perron, Oskar. 1907. Zur Theorie der Matrizen. Mathematische Annalen 64: 248-263.

Prais, S. J. 1955. Measuring Social Mobility. Journal of the Royal Statistical Society, Series A, 118, part 1: 56-66.

Rogers, Andrei. 1968. Matrix Analysis of Interregional Population Growth and Distribution. Berkeley: University of California Press.

Stone, Richard. 1966. Mathematics in the Social Sciences and Other Essays. Cambridge: Massachusetts Institute of Technology Press.

Tabah, Léon. 1968. Représentations matricielles de perspectives de population active. Population 23: 437-476.

Received October, 1974.