

"ROOTS", OR HOW A POPULATION FORGETS ITS PAST

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Résumé — Dans cette étude nous essayons de rendre plus vivant et facilement compréhensible le procédé par lequel un régime fixe de fécondité et de mortalité aboutit à une répartition par âge stable. Pour faire ceci nous considérons une variété de circonstances artificiellement simplifiées telles que les fonctions de maternité nette rectangulaires. Nous examinons la grandeur des composants oscillatoires du rang de naissance pour distinguer entre la convergence relative et la convergence absolue. Enfin, nous analysons le spectre du rang de naissance pour montrer comment il définit la structure évoluant de naissances.

Abstract — In this paper we seek to make more vivid and readily comprehensible the process by which a fixed fertility and mortality regime results in a stable age distribution. To do so we consider a variety of artificially simplified circumstances such as rectangular net-maternity functions. We examine the magnitude of the oscillatory components of the birth sequence to distinguish between relative and absolute convergence. Finally, we analyze the spectrum of the birth sequence to show how it defines the evolving structure of births.

Keywords — roots, Lotka's equation, spectrum, convergence, stability

The concepts of a stable population which were introduced by Alfred J. Lotka are very near the center of mathematical demography. The purpose of this article is to restate Lotka's concepts of stabilization in mathematical terms less advanced than he used, and then to make stabilization more understandable by exploring various extreme artificial examples that illustrate different features of the process.

The basic ideas of stable population are as follows:

(1) If a population of one sex is subject to a schedule of fertility by age that remains fixed, year after year, and subject also to a similarly fixed schedule of mortality, the sequence of births in the population ultimately comes to follow a time path in which the annual number of births changes at a fixed proportionate rate, like a sum of money at compound interest. The ultimate rate of increase of the birth sequence is wholly determined by the schedules of fertility and mortality.

(2) The age composition of the population, whatever form it may have had when the fixed regimes of fertility and mortality were initiated, ultimately comes to assume a structure — wholly determined by the fixed fertility and mortality schedules — that does not change.

(3) When the population becomes stable, with a constant geometric rate of change in the number of births, and an unchanging age distribution, it has a fixed birth rate, death rate, and rate of increase — the number of births, and the number of persons at each and every age grow at the same rate.

The inevitable tendency (with fixed fertility and mortality schedules but no matter what the initial age structure) to arrive at a fixed and determinate age structure, birth rate, death rate, and rate of increase is what is meant when it is said that a population forgets its past. Since the nature of the past (far enough back) makes no difference, we can say that it is in effect forgotten. The concepts of a stable population have been generalized to show that if fertility and mortality

schedules change rather than remaining fixed, the age composition, rate of increase, and birth and death rates also change; nevertheless, if a long enough sequence of schedules is considered, the population at the end of the sequence has an age structure and vital rates that are determined wholly by the long sequence of schedules: the structure of the population is independent of the initial conditions in the remote past. Forgetting the past is a characteristic of all populations, not just of the artificial instance of stable populations.

In restating the concepts of a stable population, we have attempted to employ the simplest possible mathematical arrangement, in the hope that it will be intuitively understandable. There is no way (not that we could devise, at least) of making the ideas transparently obvious to a person with no training whatsoever in mathematics. However, the argument presented here is designed to be readily followed by a student who has had an average course of first year college mathematics or the graduate student or older person who has had some calculus, but has not used it for some years. In the later part of the discussion more advanced mathematics are employed, and we do not expect that the mathematical competence needed for an understanding of the first part will necessarily render the arguments in the second half transparent.

Our principal aim is to clarify the logic of the argument that establishes the changing nature of the sequence of births (and of the age composition of the population) as fixed fertility and mortality bring the population to stability. We hope to make the process of stabilization more vivid and more readily comprehended by considering a wide variety of artificially simplified circumstances, much as various physical processes are understood by imagining a world of no friction, perfectly uniform fields and the like.

The path to stability.

The conditions leading to a stable population are the continued prevalence of a fertility schedule (which we will represent by $m(a)$, the proportion of women at age a who bear a female child¹), and a constant mortality schedule (which we will represent by $p(a)$, the proportion surviving from birth to age a in a population subject to the stipulated set of unchanging mortality rates). The proportion of new-born women who are subject to the stipulated regime of fertility and mortality, survive to age a and bear a daughter, is $\phi(a) = p(a) \cdot m(a)$. We may call $p(a)$ the schedule of survival, $m(a)$ the schedule of maternity, and $\phi(a)$ the schedule of net maternity.

The basic equation for the sequence of births in a population subject to the fixed regime that ultimately leads to stability is easily constructed.² Consider the female population at ages less than β at the moment the regime is initiated (where β is the greatest age at which births occur). Let us substitute for whatever number of births occurred during the β years preceding this moment the number that would yield exactly the population up to age β , had the schedule of survival $p(a)$ started at $t = -\beta$ rather than at $t=0$. In other words, to assure consistency from $t=0$, we assign to the years $t = -\beta$ to $t=0$ a number of births consistent with the observed population and the hypothetical constancy of mortality for β years before $t=0$. Then, for all t greater than zero, the number of women at age a is $B(t-a)p(a)$. The number of daughters borne by women at age a and time t is $B(t-a)p(a)m(a)$, or $B(t-a)\phi(a)$, when $B(t-a)$ is the number of births that occurred a years before time t . The total number of births at time t is the sum of the number borne by women at each age from α (the earliest age of childbearing) to β . If time and age are artificially divided into segments of one-year duration,

$$B(t) = B(t-\alpha)\phi(\alpha) + B(t-\alpha-1)\phi(\alpha+1) \dots \quad (1)$$

$$+ B(t-\beta+1)\phi(\beta-1) + B(t-\beta)\phi(\beta)$$

or,

$$B(t) = \sum_{a=-\alpha}^{\beta} B(t-a)\phi(a). \quad (2)$$

However, time and age are really continuous variables, and the conceptually correct form of (2) is in terms of an integral:

$$B(t) = \int_{-\alpha}^{\beta} B(t-a)\phi(a)da. \quad (3)$$

Equation (3) is nothing more than the expression of the sum expressed in equations (1) and (2); but equation (3) allows for the continuous nature of time and age.

Suppose we stipulate any population whatsoever under age β at time zero, and for convenience convert this arbitrary population into a sequence of births during the β years before $t=0$. Then, according to equation (3), $B(0) = \int_{-\alpha}^{\beta} B(0-a)\phi(a)da$; the births at time 0 are determined by $\phi(a)$ and the arbitrarily specified births in the interval α to β years before the initiation of the constant regime. It is not difficult to write a computer program that projects the sequence of births from this moment on by numerical approximation of the value of the integral in equation (3). For the first α years, the sequence will be determined by $\phi(a)$ and the births stipulated from $t = -\beta$ to $t=0$ (the births that would yield the initial population up to age β). Beginning with $t=\alpha$, some of the births used in constructing the projections will themselves be projected births. For t greater than β all values of $B(t-a)$ within the integral will themselves be projected values, and $B(t)$ will from then on be wholly determined by projected births. At a later point in this paper, the nature of stabilization will be illustrated by making just such projections. First, we shall outline an analytical proof that the population does stabilize.

The proof of stabilization is a demonstration that equation (3) implies that the birth sequence ultimately takes on the form:³

$$B(t) = Be^{rt} \quad (4)$$

Equation (4) is, of course, the mathematical expression for an entity growing at a constant geometric rate, a rate that may be positive (becoming ever larger), zero (remaining fixed), or negative (constantly shrinking).

Should $B(t)$ come to follow the trajectory given in equation (4) (which is one of the properties of stability), the other characteristics of a stable population are a logical consequence. The unchanging age distribution is a consequence of an exponential birth sequence: since the number at age a at time t is $B(t-a)p(a)$, and since births a years ago were changing at the constant rate r , it follows that the number of persons at every age changes at the same rate, and therefore the proportionate age distribution is fixed. The proportion at age a , $c(a)$, is the number at age a , $B(t-a)p(a)$, divided by the number at all ages, $\int_0^{\omega} B(t-a)p(a)da$. Since $B(t-a) = B(t)e^{-ra}$,

$$c(a) = \frac{B(t)e^{-ra}p(a)}{B(t)\int_0^{\omega} e^{-ra}p(a)da} = \frac{e^{-ra}p(a)}{\int_0^{\omega} e^{-ra}p(a)da} \quad (5)$$

where ω is the highest age attained. Then, since the proportion at age zero in a population is synonymous with the birth rate (b), it is seen, by letting a be zero in equation (5), and noting that $p(0)$ and e^0 are each equal to one, that $b = c(0) = 1/\int_0^{\omega} e^{-ra}p(a)da$. Equation (5) can thus be written as:

$$c(a) = b e^{-ra}p(a). \quad (6)$$

We now return to equation (3), in which births at each moment under a regime of constant fertility and mortality are related, through the net fertility schedule, to births during an interval a generation earlier. For most net fertility schedules, there is a large set — in fact, an infinite set

— of mathematical functions that satisfy equation (3). If a function does satisfy (3), and if a sequence of births — say the initial sequence calculated for the β years preceding $t=0$ — conformed to the function for β years, the function would then precisely specify the time path of births for all subsequent values of t .

There are two basic modes⁴ of behaviour of the sequence of births that can satisfy equation (3):

a) Constant geometrical growth in the number of births at a rate that is determined by $\phi(a)$: an exponential birth sequence that constantly increases, constantly decreases, or perpetually remains fixed.

b) Regular oscillations of a cycle length (or conversely, frequency of oscillation) that is determined by $\phi(a)$, oscillations that grow or diminish in amplitude at a constant rate that also is consistent with $\phi(a)$.

The first mode of behaviour is expressed in a function of the form $B(t) = Qe^{rt}$; the second mode in a function of the form $B(t) = e^{xt}(C \cos(yt) + S \sin(yt))$ or, alternatively, $B(t) = Qe^{xt} \cos(yt + \theta)$, where $\theta = -\arctan(S/C)$ and $Q = \sqrt{C^2 + S^2}$.

To show that the exponential $B(t) = Qe^{rt}$ is a solution, it is substituted in (3) with the result:

$$Qe^{rt} = Qe^{rt} \int_{\alpha}^{\beta} e^{-ra} \phi(a) da. \quad (7)$$

It is evident that equality is preserved provided r has a value that causes the integral $\int_{\alpha}^{\beta} e^{-ra} \phi(a) da$ to equal 1.0.

There is always one and only one value of r that causes the integral in equation (7) to equal 1.0,⁵ as is shown by the following argument: $\phi(a)$ is non-negative everywhere and contains positive values over some finite range. Hence, multiplication by e^{-ra} yields an integral with non-negative values, made as large as we please by selecting a sufficiently large negative value of r , and as near to zero as we please by selecting a sufficiently large positive value of r . The value of the integral decreases monotonically as r increases. Hence, it must equal 1.0 for one, and only one, value of r .

If the initial age distribution from zero to age β happened to match the survivors (according to the survival schedule $p(a)$) of births annually changing at the rate r_0 (where $\int_{\alpha}^{\beta} e^{-r_0 a} \phi(a) da = 1$), the birth sequence beginning at $t=0$ would be exactly $B(t) = B_0 e^{r_0 t}$. The requisite initial age distribution that produces this result is, naturally enough, $N(a) = B_0 e^{r_0 a} p(a)$; in short, the stable age distribution. In other words, if the initial age distribution has the stable form, the birth sequence follows an exponential path from the outset, and stability of the population is maintained throughout.

Now consider the second mode of behaviour of a birth sequence that satisfies equation (3), namely $B(t) = e^{xt}(C \cos(yt) + S \sin(yt))$. Substituting in equation (3), we obtain:

$$\begin{aligned} e^{xt}(C \cos(yt) + S \sin(yt)) &= \int_{\alpha}^{\beta} e^{x(t-a)} C \cos(y(t-a)) \phi(a) da \\ &+ \int_{\alpha}^{\beta} e^{x(t-a)} S \sin(y(t-a)) \phi(a) da \end{aligned} \quad (8)$$

But $e^{x(t-a)} = e^{xt} e^{-xa}$; $\cos(y(t-a)) = \cos(yt)\cos(ya) + \sin(yt)\sin(ya)$; and $\sin(y(t-a)) = \sin(yt)\cos(ya) - \cos(yt)\sin(ya)$.⁶ Hence

$$\begin{aligned} e^{xt}(C \cos(yt) + S \sin(yt)) &= e^{xt} C \cos(yt) \int_{\alpha}^{\beta} e^{-xa} \cos(ya) \phi(a) da \\ &+ e^{xt} C \sin(yt) \int_{\alpha}^{\beta} e^{-xa} \sin(ya) \phi(a) da \\ &+ e^{xt} S \sin(yt) \int_{\alpha}^{\beta} e^{-xa} \cos(ya) \phi(a) da \\ &+ e^{xt} S \cos(yt) \int_{\alpha}^{\beta} e^{-xa} \sin(ya) \phi(a) da. \end{aligned} \quad (9)$$

The balance of equation (9) is assured if and only if:

$$\int_{\alpha}^{\beta} e^{-x a} \cos(y a) \phi(a) da = 1, \text{ and} \quad (10)$$

$$\int_{\alpha}^{\beta} e^{-x a} \sin(y a) \phi(a) da = 0.$$

In general there are an infinite number of values of (x, y) that satisfy equation (10). The smallest value of y always lies between $\frac{2\pi}{\beta}$ and $\frac{2\pi}{\alpha}$; the average spacing between consecutive values of y over long intervals is $\frac{2\pi}{\beta}$. We shall illustrate the characteristics of $\phi(a)$ that determine the values of (x, y) in a wide range of simplified schedules at a later point. (Coale, 1972; Coale and McNeil, 1976; Turner, 1976.)

Earlier, we saw that if the initial population happened to match the stable age distribution up to age β the birth sequence would follow an exponential sequence from $t = 0$. Is there an initial age distribution that would yield a damped sinusoidal sequence from $t = 0$? If the initial population from age zero to age β happened to match the survivors (assuming the survival schedule was $p(a)$) of births that had followed a sequence of $Q e^{x t} \cos(y t + \theta)$ during the β years preceding the initiation of the constant regime, and if x and y satisfy equation (10), the birth sequence at all times after $t = 0$ would be exactly $B(t) = Q e^{x t} \cos(y t + \theta)$. However, to imagine such an initial population and such a birth sequence is a considerable feat for the reader, since half the time $B(t)$ must be negative. We must imagine that negative women, subject to the net fertility function $\phi(a)$, generate negative births. The independent existence of the second mode of behaviour of the birth sequence is a physical impossibility — a mathematical construct with no real meaning.

Let us now consider a physically realizable initial population that includes oscillations in its age distribution and generates oscillations in the subsequent sequence of births. Consider a population that is the sum of two components, a component, $Q_0 e^{-r a} p(a)$, that would generate solely an exponential birth sequence (growing, remaining constant, or shrinking according to whether r is greater, equal to, or less than zero) and a component, $Q_1 e^{-x a} \cos(-y a + \theta) p(a)$, that would generate solely growing, shrinking, or constant amplitude cyclically varying numbers of births, with a period of $(2\pi/y)$ years. Suppose that r satisfies equation (3) and x and y satisfy equation (10) with the given $\phi(a)$.

The birth sequence of this population would be $B(t) = Q_0 e^{r t} + Q_1 e^{x t} \cos(y t + \theta)$. We have calculated r , x , and y for two interrelated net fertility schedules: schedules with exactly the same age pattern (chosen to be typical of human experience), but differing in aggregate area, since each schedule has been multiplied by a different constant, so that in one instance $r = 0$ and $x < 0$, while in the second instance $r > 0$ and $x = 0$. (The calculated value of y is the smallest among the infinite array of possibilities.) The value of Q_0 has been set at 1,000,000 and Q_1 has been given the largest value that avoids the impossible implication of negative numbers at some ages in the initial population. The initial population for each net fertility schedule is shown in Figure 1 as the sum of its two components. Figure 2 shows the resultant birth sequences ($Q_0 e^{r t}$, $Q_1 e^{x t} \cos(y t + \theta)$, and the sum of these two components). Note that when r is zero, so that the non-oscillatory component is constant at 1,000,000 births a year, the oscillatory component is multiplied by a diminishing exponential, so that the oscillatory term rapidly shrinks, and becomes negligible after a few generations. At a much higher level of fertility, such that the oscillatory component maintains a constant amplitude, the non-oscillatory component increases quite rapidly. The oscillations still become negligible as a fraction of total births. In both instances, the birth sequence *converges* to a pure exponential sequence — when r is moderate, x is negative, and the oscillatory component tends to zero. (If r were negative, x would be *more* negative.) The reduction to negligible magnitude of the oscillations is *absolute convergence* of the birth sequence. When x is zero or positive, it is the *ratio* of the oscillations to pure exponential that becomes a negligible fraction, a process that can be called *relative convergence*.

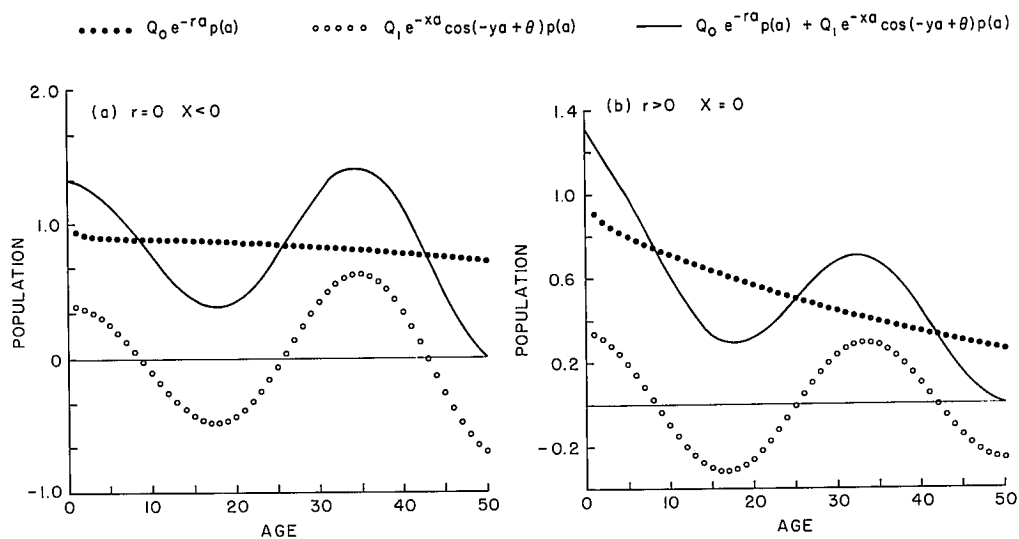


FIGURE 1. POPULATION RESULTING FROM EXPONENTIAL AND OSCILLATORY COMPONENTS AND THEIR TOTAL

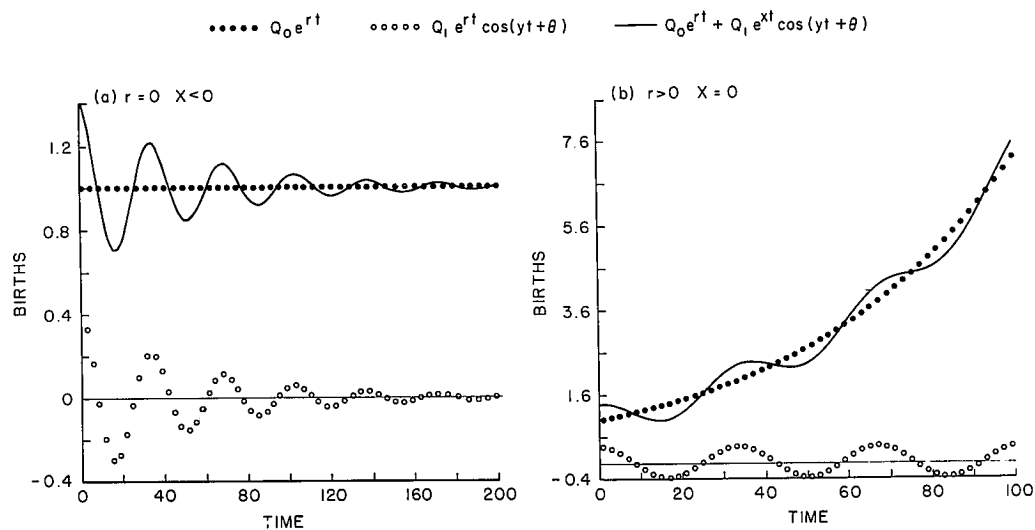


FIGURE 2. BIRTHS RESULTING FROM EXPONENTIAL AND OSCILLATORY POPULATION COMPONENTS AND THEIR TOTAL

Figure (2) illustrates, then, the convergence of population to a stable form with two similarly structured net fertility schedules at different levels of fertility, and with very special initial populations that generate only a single oscillatory component of the birth sequence. The illustrations incorporate a crucial feature that insures the convergence to stability of actual populations subject to realistic net fertility schedules: the value of r (the uniquely determined exponent in the pure exponential part of the birth sequence) is always greater than the value of x , the exponent of the exponential multiplier of the sinusoidal term in any oscillatory component. Thus the exponential component of the birth sequence always grows faster or shrinks slower than any oscillatory component, so that eventually every oscillatory component becomes a negligible fraction (by any predetermined criterion of negligible, such as less than one per cent) of the total number of births. When all oscillatory components are negligible, the birth sequence may be considered purely exponential, and the critical criterion for stability has been met. The age distribution becomes fixed no more than ω years after the births sequence is exponential.

There are two points to be added to complete this description of stabilization. First, the birth sequence generated by any arbitrary initial population containing a large number of women at ages below β can be expressed, for all times after the initiation of constant fertility and mortality, as the sum of one pure exponential component, and a series of oscillatory components each multiplied by an exponential. In mathematical terms:

$$B(t) = Q_0 e^{rt} + Q_1 e^{x_1 t} \cos(y_1 t + \theta_1) + Q_2 e^{x_2 t} (\cos y_2 t + \theta_2) + \dots \quad (11)$$

or

$$B(t) = Q_0 e^{rt} + \sum_{i=1}^{\infty} Q_i e^{x_i t} \cos(y_i t + \theta_i) \quad (12)$$

Second, it can be proved that r (the exponent of the one purely exponential component) is always greater than x_i (the exponent of the multiplier of the i^{th} oscillatory component).

In equation (12) the value of r is that which satisfies $\int_{\alpha}^{\beta} e^{-ra} \phi(a) da = 1$, and each x_i and y_i satisfy $\int_{\alpha}^{\beta} e^{-x_i a} \cos(y_i a) \phi(a) da = 1$ and $\int_{\alpha}^{\beta} e^{-x_i a} \sin(y_i a) \phi(a) da = 0$. The rate of increase, r , the exponents, x_i , of the exponential multipliers of the oscillatory components, and the period, $2\pi/y_i$, of the oscillatory components thus are wholly determined by the net maternity function $\phi(a)$. From $t = 0$ to $t = \alpha$ (α the earliest age of childbearing) all births occur to women in the arbitrary initial age distribution, and until $t = \beta$ births to mothers in the initial population continue. In order that $B(t)$ expressed by the components of equation (10) should fit the early part of the birth sequence, the oscillatory components must have amplitudes and phases so that their additions (when all have the same sign), or partial cancellation (when some are positive and some are negative) combine to match the irregularities in the early birth sequence caused by peculiarities of the initial age distribution. Thus the values of Q in equation (12) are jointly determined by the net maternity function and the initial conditions.⁷ In other words, the nature of $\phi(a)$ determines what functions are consistent with the relation $B(t) = \int_{\alpha}^{\beta} B(t-a) \phi(a) da$ (the permissible functions are generally infinite in number); the initial conditions then determine the magnitude (and phase, for the oscillatory terms) of the constituent functions. The oscillatory terms combine with the purely exponential term to create the irregular early sequence caused by the interaction of the arbitrary age distribution and the newly fixed $\phi(a)$. With the passage of time the oscillatory terms all diminish in amplitude, either absolutely toward zero, or at least relatively compared to the pure exponential, so that ultimately only the exponential term is significant. The birth sequence thus becomes exponential; the arbitrary initial conditions are forgotten.

The oscillatory terms become negligible because the value of r is greater than x_i , for every

oscillatory term. This relation follows from two equations, one that r must satisfy, and one that each x must satisfy. These are, respectively, $\int_{\alpha}^{\beta} e^{-ra} \phi(a) da = 1$ and $\int_{\alpha}^{\beta} e^{-xa} \cos(ya) \phi(a) da = 1$. Note that if $\cos(ya)$ were 1.0 for all values of a from α to β the two equations (one involving r and the other x) would be the same, and r and x would be equal. $\cos(ya)$ can equal 1.0 only at a finite number of points within the interval from α to β (whenever $ya = 2n\pi$), and is less than one at all other points. Since $\phi(a)$ for human populations is a continuous positive function over some finite range of ages, $\cos(ya)\phi(a)$ must be never greater than $\phi(a)$ and less than $\phi(a)$ at some points. Therefore if x were equal to r , $\int_{\alpha}^{\beta} e^{-xa} \cos(ya) \phi(a) da$ would be less than 1.0; to have the integral incorporating the cosine term equal to 1.0 requires that the exponential factor e^{-xa} be greater than e^{-ra} , and since x and r are both associated with minus signs, x must be less than r .

Limiting forms of $\phi(a)$, a) bringing instant convergence of the birth sequence, and b) never bringing convergence, to an exponential birth sequence

The feature of fertility and mortality schedules that leads ultimately to a smooth exponential sequence is that $\phi(a)$ spreads the childbearing of a cohort of women over a range of ages. The births occurring at a given moment can in consequence be viewed as the product of (1) a measure of the overall level of net fertility, and (2) a weighted average of the number of births α to β years earlier, the weights being the values of $\phi(a)$. How averaging occurs is made explicit by rewriting the basic equation of reproduction with fixed schedules, $B(t) = \int_{\alpha}^{\beta} B(t-a)\phi(a)da$, in this form:

$$B(t) = R \int_{\alpha}^{\beta} B(t-a)f(a)da, \quad (13)$$

where $R = \int_{\alpha}^{\beta} \phi(a)da$, the net reproduction rate (or the average number of daughters per woman when subject to the maternity function and survival function that define $\phi(a)$); $f(a)$ is $\phi(a)/R$, or the proportionate distribution of net fertility by age so that $f(a)$ has the properties of a frequency distribution. $B(t)$ is R times the weighted average (weights given by $f(a)$) of the births α to β years earlier. If the initial population has a highly irregular age distribution — exemplified, let us say, by the presence of an unusually small cohort — the effect on later births (i.e., a later dip caused by the small initial cohort) is attenuated by the averaging of births to the small cohort with births of other cohorts of more normal size that are in the childbearing span at the same time. The forces of convergence are quickly effective to the extent that this averaging of births occurring to different cohorts quickly erases initial irregularities.

There are two extreme hypothetical forms of $\phi(a)$ with respect to rapidity of convergence: one leads immediately to a purely exponential birth sequence no matter what the initial age distribution, and the other preserves indefinitely every irregularity in the initial distribution (up to age β) without any reduction as time passes in the magnitude of fluctuations relative to the average level of the number of births.

The schedule of net fertility that produces instant convergence in the birth sequence is a $\phi(a)$ that is itself a declining exponential function of age. Such a function is implied if $m(a) = m$ at all ages, and the death rate is k at all ages. A death rate that does not vary with age implies $p(a) = e^{-ka}$; and if $m(a) = m$, it follows that $\phi(a) = me^{-ka}$. But if fertility is m at all ages, and mortality is k at all ages, the number of births is a simple multiple, m , of the total number of persons, and the number of deaths is, likewise, k times the number of persons. The number of persons, and the number of births, thus grow at the constant rate, $r = m - k$, no matter what the age distribution.⁸ The past is instantaneously forgotten.

The other extreme of a schedule of net fertility that perpetuates initial irregularities is $\phi(\beta) = R$ (the net reproduction rate), and $\phi(a) = 0$ at all other ages.⁹ The equation $B(t) = \int_{\alpha}^{\beta} B(t-a)\phi(a)da$ now becomes $B(t) = R \cdot B(t-\beta)$. If we reconstruct a fictitious series of births (from

$t = -\beta$, to $t = 0$, such that births at $t = -a$ are $N(a)/p(a)$, that would have led to the initial age distribution, it can be seen that $B(0) = R \cdot N(\beta)/p(\beta)$, $B(1) = R N(\beta-1)/p(\beta-1)$, and $B(\beta-1) = R N(1)/p(1)$, so that whatever irregularity existed in the initial population is exactly duplicated (scaled upward or downward by the factor R) in the births from $t = 0$ to $t = \beta$, and will be duplicated again (scaled by the factor R^2) in the period from $t = \beta$ to $t = 2\beta$. Indeed, births after $t = 0$ are made up of independent projections of the initial population at different ages, with the population at age (a) producing future births at time $t = \beta - a, 2\beta - a, 3\beta - a$, etc., of number $R \frac{N(a)}{p(a)}$, $R^2 \frac{N(a)}{p(a)}$, $R^3 \frac{N(a)}{p(a)}$. Clearly this set of independent sequences preserves precisely the initial irregularities. This population does not forget the past; it remembers it exactly.

Consider the components (purely exponential and exponentially modified oscillations) of $B(t)$ with this net fertility function $\phi(a) = R$ when $a = \beta$, and zero at other ages). The pure exponential component is determined by the equation $\int_0^\beta e^{-ra} \phi(a) da = 1$, which becomes $R e^{-r\beta} = 1$; hence $r = \log R/\beta$.¹⁰ For each oscillatory component the equations, $\int_0^\beta e^{-xa} \cos(ya) \phi(a) da = 1$ and $\int_0^\beta e^{-xa} \sin(ya) \phi(a) da = 0$, become $R e^{-x\beta} \cos(y\beta) = 1$, and $R e^{-x\beta} \sin(y\beta) = 0$. These equations are satisfied for values of $y\beta$ that are multiples of 2π , because only at multiples of 2π is the sine zero and the cosine positive. Hence, $y = n \cdot \frac{2\pi}{\beta}$, $n = 1, 2, 3, \dots$. Since $\cos(n \cdot 2\pi) = 1$, $R e^{-x\beta} = 1$, or $x = \log R/\beta$, precisely equal to r . Thus the oscillatory components have frequencies of one cycle every β years, 2 cycles every β years, etc., and each oscillatory component is multiplied by e^{xt} , where $x = r$. The functional representative of $B(t)$, then, is:

$$B(t) = e^{\frac{\log R}{\beta} t} (Q_0 + \sum_{i=1}^{\infty} Q_i \cos(\frac{2\pi i}{\beta} t + \theta_i)) \quad (14)$$

All of the oscillatory terms (at the frequency of $1/\beta$ and all of its harmonics) grow or shrink at the same rate ($r = \log R/\beta$) as the one non-oscillatory term. The ratio of the amplitude of each oscillatory component to the one non-oscillatory component, rather than decreasing until it becomes negligible, remains fixed. The values of Q_0 and of the various Q_i 's and θ_i 's are calculated to match the irregularities of the birth sequence generated by the initial population, and the precisely equal value of r and all of the x 's duplicates these irregularities (multiplied by R, R^2, R^3 , etc.) in each subsequent interval of β years.

Convergence to stability when the schedule of net fertility is a rectangle of various widths

It was noted earlier that the basic mechanism causing a birth sequence to converge to an exponential when net fertility is constant is the averaging (with weights proportional to $\phi(a)$) of numbers of births α to β years ago in determining the number of births now occurring. This averaging progressively reduces the effect on later births of any initial irregularity such as an unusually large or small birth cohort. To illustrate how the characteristics of $\phi(a)$ determine the way births vary in number until convergence occurs, we shall consider an unrealistic but very simple set of net fertility functions, of a form that provides the simplest kind of averaging — equal weights over a specific interval. This artificially simple function is constant net fertility over a range from α to β , or $\phi(a) = R/(\beta - \alpha)$, for $\alpha \leq a \leq \beta$, where R is the net reproduction rate. With such a fertility function, the births at time t are a constant times the arithmetical (equally weighted) average of the births α to β years earlier. We shall consider the effect of variations in the start and finish of such rectangular $\phi(a)$'s (or variations in the width of the interval from α to β). This task is simplified because all genuinely different possibilities are exhausted by holding β fixed at any arbitrary number (one or ten, for example), and allowing α

to vary from zero to just short of β . All other rectangular fertility schedules are simply scaled up or scaled down replicates of one of the schedules constructed in this way. For example, with a given net reproduction rates, a schedule extending from 1.5 to 4.0 has properties that can be immediately translated to a schedule that extends from 18 to 48. Multiplication of all ages by 12 is precisely equivalent to measuring time in months instead of years. The result is to reduce r and all of the x 's and y 's that characterize the birth sequence by a factor of 12 because growth per month (or the number of cycles per month) is one-twelfth the corresponding annual figure. We shall accordingly consider β fixed at a convenient figure (one or ten), and allow α to vary from zero to just short of β .

If the net reproduction rate is set equal to 1.0, so that the ultimately stabilized birth sequence is constant ($r=0$), the slowest rate at which any initial irregularities disappear is indicated by the largest algebraic value of x_i (the exponent of the multiplier of the i^{th} oscillatory component). When $r=0$, the largest value of x_i is always associated with the lowest frequency (longest period) oscillatory term, as α varies from zero to just short of ten, where β is always set at ten. The values of x_1 to x_5 , when $r=0$, are shown in Figure 3, for this set of rectangular net fertility functions. The value of x_1 — the rate at which the “most favoured” oscillatory term

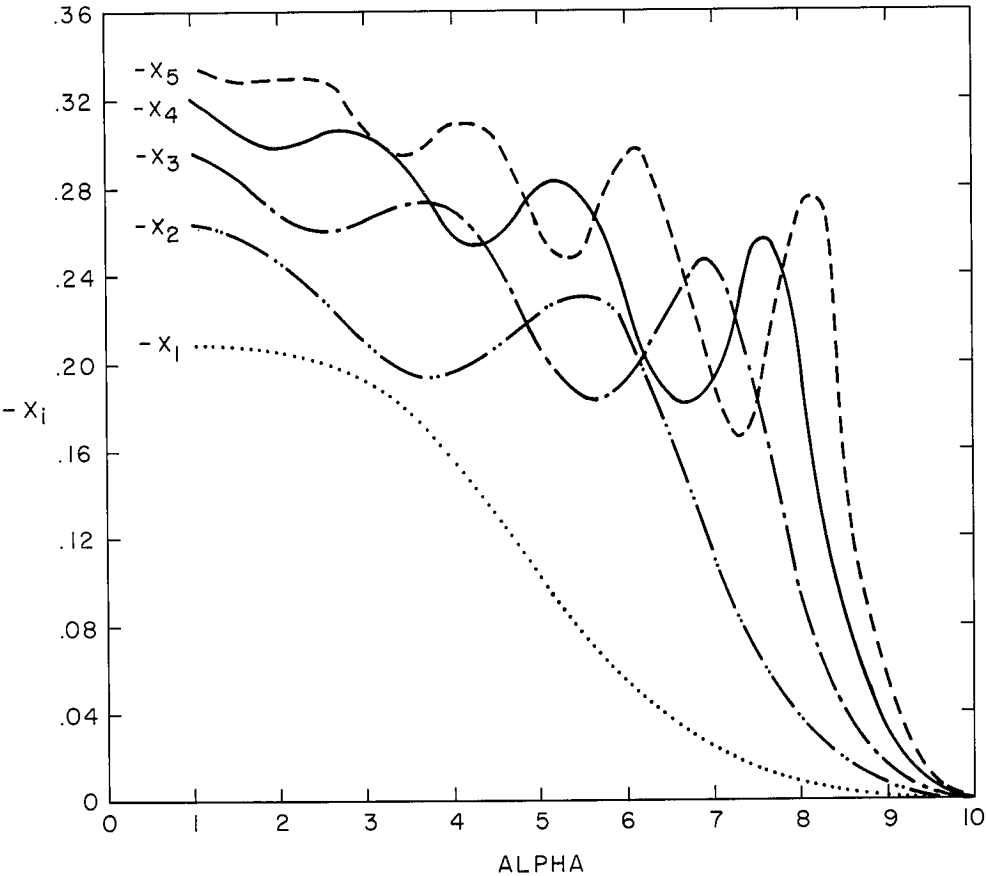


FIGURE 3. PLOT OF THE REAL PARTS OF THE FIRST FIVE COMPLEX ROOTS AGAINST ALPHA

diminishes relative to the underlying horizontal trend — bears a smooth and qualitatively understandable relation to the width of the rectangular $\phi(a)$. When $\alpha=0$ (maximum width, fertility begins at birth), the range of past births that are averaged in determining current births is at its greatest extent; the value of x_i is about -0.208 , implying a 21 per cent annual decline in the amplitude of the longest period oscillatory component, or an 88 per cent reduction every β years ($e^{-10(0.208)}=0.12$). The value of x_i is much the same for all rectangular $\phi(a)$'s where α is no greater than 10 per cent of β but it rises (toward zero), becoming quite close to zero (greater than -0.002) when α is 9 ($\beta=10$). The greater the interval of ages of women who contribute to births at each moment, the faster low frequency oscillations are erased. The ‘damping’ of oscillatory terms of shorter periods (higher frequency) is related to α in a more complicated way, and will be discussed only at a later point.

The corresponding values of the periods of the oscillatory terms are shown in Figure 4, specifically the values $2\pi/y_1, 4\pi/y_2, \dots, 10\pi/y_5$ are shown, where the period $2\pi/y_i$ has been multiplied by i so that all can be shown in the same figure. The solid straight line shows the value at the midpoint of the rectangle which extends from α to 10. If the period of the first oscillatory term were equal to the midpoint of the interval, and the period of the second term were equal to half the midpoint, etc., then all values in Figure 4 would lie on this line. Scrutiny of the plot will show that the higher is α the closer the values of $2\pi i/y_i$ are to the line.

The convergence of the birth sequence to an exponential sequence (here a constant trend, with $r=0$) with two limiting forms of rectangular net fertility functions will now be considered in more detail. The two extremes are first a net fertility function that begins at age zero and

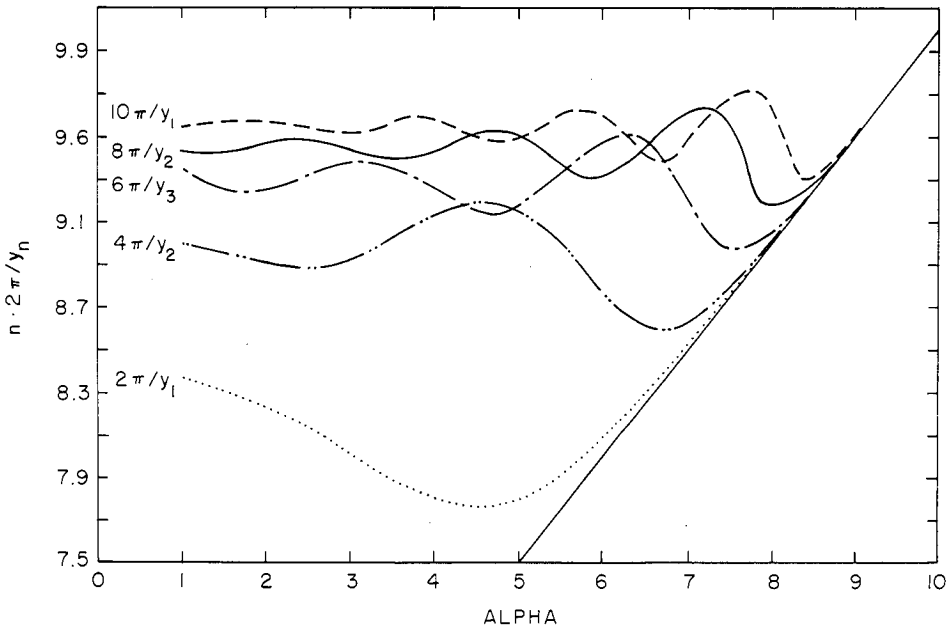


FIGURE 4. PLOT OF THE PERIODS OF THE FIRST FIVE COMPLEX ROOTS AGAINST ALPHA. THE PERIODS HAVE BEEN MULTIPLIED BY AN APPROXIMATE SCALAR

extends to age ten, and second, a function that begins at 9.0 and extends to ten. These limiting cases of rectangular functions approach the true limiting cases of $\phi(a)$ discussed earlier: i.e. when fertility begins at age zero, it has much in common with the negative exponential $\phi(a)$ that produces instant stabilization; when $\phi(a)$ is concentrated near β it approaches the situation in which reproduction occurs at a single age, wherein there is no convergence. Convergence will be compared by examining the birth sequence that would be generated if the initial population were a hypothetical group of women just born at the same instant. This initial population (called by Lotka a "population element") provides a birth sequence that is particularly easy to visualize. A population element is the greatest imaginable departure from a stable birth sequence; or, in other terms, it maximizes the initial magnitude of the oscillatory components relative to the one pure exponential component, and takes the longest time to stabilize of all initial populations.

Slow convergence when net fertility extends from 9.0 to 10.0

If net fertility were a rectangle from 9.0 to 10.0, with a net reproduction rate of 1.0, and the initial population were an element at age zero (of, let us say, one million female births), for many years the different generations of descendants of the initial cohort would occur in non-overlapping time intervals. Let $B_0(t)$ be the initial cohort, the zero order generation, all born at $t=0$. The first generation consists of women borne by the original cohort, the second generation of women borne by the first generation, and so on. The births of the first generation, $B_1(t)$, occur when the zero order generation is at an age t between α and β ; or $B_1(t) = B_0 \cdot \phi(t)$. When second generation births are occurring, at some later time t years after the birth of the initial cohort, each birth must be to a first generation woman at some age a between α and β , whose mother of the zero order generation, born at $t=0$ was necessarily at age $t-a$ at the time the first generation woman was born. In other words, when a second generation child is born at time t , the sum of the age at maternity of the mother who bore the child and of the age at maternity of the grandmother who bore the mother must be t . First generation births occur between the limits of $t=\alpha$ and $t=\beta$; n^{th} generation births between $n\alpha$ and $n\beta$. The distribution through time of $B_1(t)$ is $B_0 \phi(t)$. The distribution of the second generation births is governed by the relation $B_2(t) = \int_{\alpha}^{\beta} B_1(t-a)\phi(a)da$. The distribution of second generation births in time measured from the birth of the zero generation cohort ($t=0$) is the distribution of a variate that is the sum of two other independently distributed variates (the age at maternity of the two mothers in generations zero and one). Both distributions are $\phi(a)$, the unchanging net maternity function. The distribution of the sum of two independently distributed variates is called the *convolution* of the distribution; the distribution of $B_n(t)$ is the convolution of $\phi(a)$ with itself n times, called the n -fold convolution.

The distribution of $B_n(t)$ (the n -fold convolution of $\phi(a)$) has a simple interpretation: it differs only in scale and location from the distribution of the mean age at birth of a sample of n women who give birth according to $\phi(a)$. The variate that defines $B_n(t)$ is the mean age at birth ($a_1 + a_2 + \dots + a_n$) of n women each subject to $\phi(a)$; the average age at birth of these n women is $\frac{a_1 + a_2 + \dots + a_n}{n}$. The distribution of $B_n(t)$ and of the mean age at maternity of a sample of n women subject to $\phi(a)$ are thus the same in form, but the convolution is on a scale n times as great as the distribution of sample means. It follows that the mean time of occurrence of births in the n^{th} generation is n times the mean age of $\phi(a)$ and that the standard deviation is the standard deviation of $\phi(a)$ multiplied by n .¹¹

As successive generations are born, the width of the time interval during which the births of each generation occur increases. When $\phi(a)$ is rectangular from α to β , the mean time of birth for the n^{th} generation is $(n/2)(\alpha + \beta)$; the time from the earliest to the latest births extends from $n\alpha$ to $n\beta$. Thus when n is greater than $\alpha/(\beta - \alpha)$, the births of successive generations overlap.

However, the standard deviation of the distribution through time of births of the n^{th} generation is the standard deviation of $\phi(a)$ times n , or $n(\beta - \alpha)/\sqrt{12}$. The distribution of $B_n(t)$ approaches a normal distribution as n increases (a well-known characteristic of the distribution, as n increases, of the mean of a sample drawn from any distribution with a finite variance),¹² so that more than 2/3 of the births of the n^{th} generation (for n greater than five or six) lie within a standard deviation of the mean. If $\alpha=9$ and $\beta=10$, the standard deviation of the distribution of $B_n(t)$ is $n/\sqrt{12}$. The interval between the mean date of birth in successive generations is 9.5 years; not until $n/\sqrt{12}$ equals about 9.5/2 (i.e. $\sigma = 9.5/2$) do the latest 1/6 of n^{th} generation births overlap with the earliest 1/6 of the $(n + 1)^{\text{st}}$ generation—at about the sixteenth generation. Ultimately, the births at any moment include members of a large number of overlapping generations (births in each generation widely spread, in a normal distribution with a large standard deviation). The declining numbers in generations with births centred earlier than the given moment come to be exactly offset by rising numbers in generations with births centred later than the given moment. When sixteenth generation births occur, at about $t=152$ years, the overlap of births in successive generations is about 33 per cent and the variation in births is still very substantial.

Since $B_n(t)$ is the n -fold convolution of $\phi(a)$ the sequence of births is not difficult to visualize. The number of births at each moment is always proportional to the total number that occurred in the interval nine to ten years earlier. In the first generation the process can be visualized as a population "spike" or element passing through a rectangular fertility schedule. Hence, $B_1(t)$ is a rectangle from nine to ten. In the second generation, second generation births can be visualized as resulting from a population rectangle (the first generation births) passing through a rectangular fertility schedule. The birth sequence must rise and then fall since the overlap between the population rectangle and the fertility rectangle rising reaches a maximum at $t=19$, and thereafter falls. Indeed, starting at $t=18$, the number of births nine to ten years earlier rises linearly until, at $t=19$, all of the first generation births occurred from nine to ten years earlier; then from $t=19$ to $t=20$, the number of births declines linearly to zero. Third generation births arise as a population triangle (second generation births) passes through a rectangular fertility schedule. It can be shown that from year 27 to 28 third generation births follow a second degree curve, $(t-27)^2/2$; from 28 to 29 follow another second degree curve $(1-2(t-28.5)^2)$ that is concave downward, reaching a maximum at 28.5; and from 29 to 30, a third second degree segment, $(t-30)^2/2$ that falls to zero by the 30th year. Note that the smooth and symmetric sequence of third generation births already has a structure not unlike the normal distribution.

We shall now consider the representation of this very slowly converging births series, with its changing time pattern from generation to generation, as the sum of a pure exponential (in this case with $r=0$) and an infinite number of oscillatory (sinusoidal) components, all of constantly diminishing amplitude.

The "spectrum" of a birth sequence that converges very gradually

The birth sequence of any population subject to unchanging fertility and mortality (including the very slowly stabilizing sequence when $\phi(a)$ is a rectangle from nine to ten) is made up in part of the sum of oscillatory terms at different frequencies of oscillations that diminish in amplitude (they must do so, at least, when r is zero, as in the example we are considering). We shall see that in a slowly converging birth sequence over an interval in the neighborhood of any given moment during the convergence to stability, the variations in births that occur about the long range exponential trend (here a constant trend) are determined, to a good approximation, by the amplitude and phase of the oscillatory components at the moment in question. The qualitative notion that we wish to illustrate is how the changing nature of the sequence is related to the changing composition of the oscillatory terms. Gradual variation in the number of births

connotes the prominent presence of low frequency oscillations, and sharp variation — for example, virtually instantaneous increases or decreases in the number of births, or any sharp corners, such as sudden reversals in the slope of the sequence — connote the prominent presence of high frequency terms.

In this section we will show that the changing relative prominence of components at different frequencies — the changing “spectrum” of the birth sequence — is consistent with (indeed, defines) the evolving structure of the births generated by a population that originates with an initial element at age zero, and has a fixed net fertility function, with NRR equal to 1.0, rectangular from age nine to ten.

We begin by defining the “spectrum” of a sequence of births, of arbitrary time structure $b(t)$ over any interval from t_1 to t_2 , as the amplitude and phase of all members of a set of cosine functions that duplicate $b(t)$ over the interval in question. For the spectrum to be completely specified $b(t)$ must be part of a cyclical sequence that repeats itself exactly every T years. To determine an illustrative exact spectrum of the rectangular burst of first generation births from year nine to year ten, we shall assume, for the moment, that this burst of births is part of a fully periodic sequence, in which the births of each generation are all rectangular bursts of one year’s duration, rather than evolving in form from generation to generation as do the births in the actual projection.

It is possible to represent any repetitious function of time as $f(t) = \frac{Q}{2} + \sum_{i=1}^{\infty} (C_i \cos(\frac{2i\pi}{T} t) + S_i \sin(\frac{2i\pi}{T} t))$ where Q is the average value of the function over the cycle, and T is the duration of the cycle.¹³ $1/T$ is called the fundamental frequency, and frequencies of i/T for i greater than one constitute the harmonics. Alternatively, $f(t)$ can be expressed as

$$f(t) = \frac{Q_0}{2} + \sum_{i=1}^{\infty} Q_i \cos(\frac{2i\pi}{T} t + \theta_i) \quad (15)$$

where $Q_i = \sqrt{C_i^2 + S_i^2}$ and $\theta_i = -\arctan(S_i/C_i)$. The “spectrum” is the set of C_i ’s and S_i ’s (or alternatively of Q_i ’s and θ_i ’s) that duplicate $f(t)$. The spectrum can be represented by a plot of amplitude (and phase) against frequency. The spectrum is wholly implied by the value of $f(t)$ over a cycle, and *vice versa*.

The rules for determining the spectrum of a periodic function are:

$$\begin{aligned} C_i &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(\frac{2i\pi}{T} t) dt \\ S_i &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(\frac{2i\pi}{T} t) dt \\ Q &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) dt \end{aligned} \quad (16)$$

The spectrum of a positive rectangle of height H and width W that is repeated every T years is particularly easy to determine from equations (16). For convenience, t is set equal to zero in the middle of one of the rectangles. Since the sequence is zero over the part of the cycle not within the rectangle, the limits of integration can be changed to $-W/2$ and $+W/2$; $f(t)$ can then be represented by the constant H . Clearly, $Q = 2HW/T$. Since $\sin(x) = -\sin(-x)$, the value of the sine from $-W/2$ to zero is equal and opposite to the value from zero to $W/2$ for each of the

sine functions; hence all S_i are zero. Since $\cos(x) = \cos(-x)$, the equation for C_i can be written $C_i = \frac{4H}{T} \int_0^{W/2} \cos\left(\frac{2i\pi}{T}t\right)dt$. Integrating, we obtain $C_i = \frac{4H}{T} \cdot \frac{1}{(2i\pi)/T} \left(\sin \frac{i\pi W}{T}\right)$ or $C_i = 2 \left(\frac{HW}{T}\right) \left(\frac{\sin(i\pi W/T)}{(i\pi W/T)}\right)$. If we consider the spectrum in terms of Q_0 , Q_i , and θ_i , $Q_0 = \frac{2HW}{T}$, $Q_i = C_i = \frac{2HW}{T} \left(\frac{\sin(i\pi W/T)}{i\pi W/T}\right)$, and $\theta_i = 0$. Note that $Q_i/Q_0 = \sin(i\pi W/T)/(i\pi W/T)$, and that when i/T (the frequency of the i^{th} harmonic) is between 0 and $1/W$, the value of Q_i/Q_0 is positive, from $1/W$ to $2/W$ the value is negative, and so on, with alternating sign over ranges of frequency of width $1/W$. In the hypothetical cyclical birth sequence we are considering, $H = B_0 = 1$, $W = 1$, and $T = 9.5$. The resultant Q_i/Q_0 is plotted against frequency (i/T) in Figure 5. The spectrum of this sequence is thus $\sin(x)/x$ for x defined at $\pi W/T$, $2\pi W/T$, etc. If the periodic rectangular burst of births remained the same in height and width, and the cycle length doubled, the shape of the spectrum would be unaffected; there would simply be twice as many frequencies within the range from zero to $1/W$. The “envelope” of the Q_i 's as a function of frequency would always change sign at $f = 1/W$, $2/W$, etc. and its form would always be $Q(f)Q_0 = \sin(\pi f W)/(\pi f W)$. Thus the basic nature of the spectrum of a repeated isolated rectangle depends only on its width, not on the duration of the cycle of its repetition.

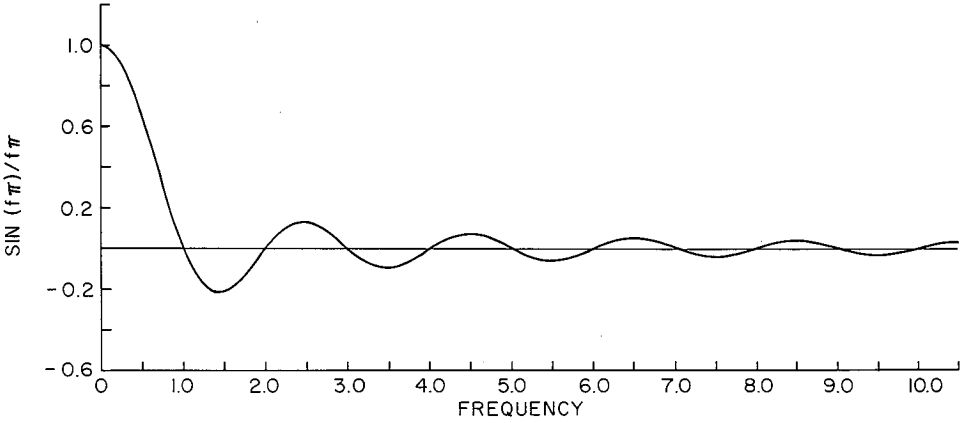


FIGURE 5. PLOT OF $\text{SIN}(f\pi)/f\pi$ vs FREQUENCY f

We shall now compare the spectrum of a repeated isolated rectangle with the frequency composition at $t=9.5$ of the birth sequence generated by a rectangular net fertility function, net reproduction rate one, extending from age nine to age ten, when the initial population is an element at age zero. For a 9.5 year time period that is centred on $t=9.5$, this birth sequence is exactly the same as a single cycle of the hypothetical infinite repeated set of one-year rectangles, for which we have just calculated the spectrum. The non-cyclical birth sequence also has a spectrum, or a pseudo-spectrum, at $t=9.5$. Recall that the non-cyclical sequence is expressed as

$$B(t) = Q_0 + \sum_{i=1}^{\infty} Q_i e^{x_i t} \cos(y_i t + \theta_i) \quad (17)$$

The x 's and y 's are calculated from the rectangular net fertility function, and the Q 's from the stipulated initial population in conjunction with $\phi(a)$. Note that equation (17), for a birth sequence that changes in form every generation, is the same as equation (15) for a wholly periodic time series, except that (17) includes the factor $e^{x_i t}$ multiplying each cosine term.

Suppose that the time origin of this sequence is shifted to the middle of the first generation births, at $t=9.5$ on the original time scale. $B(t)$ with t measured from the new origin is still specified by equation (17) with the same Q_0 , and the same x 's and y 's; however, the new Q_i 's and θ_i 's are the old Q_i times $e^{9.5x_i}$ and the old $\theta_i + 9.5y_i$.

The spectrum can be represented visually as a plot of amplitude vs. frequency, where amplitude is measured as a positive value. In figure 6 the value of $|Q_i| e^{9.5x_i}$ (the amplitude at $t=9.5$ of the i^{th} oscillatory term) is plotted against frequency ($y_i/2\pi$ cycles per year), along with $|Q_0| \sin(\pi f)/(\pi f)$, the spectrum of a repeated rectangle centred at the same point as the rectangle of first generation births. The "pseudo-spectrum" at $t=9.5$ of the non-periodic birth sequence is virtually identical with the genuine spectrum of a repeated one-year wide rectangle, not perfectly identical, to be sure, since the amplitudes of the constituent cosine functions are constantly diminishing rather than remaining the same. The amplitudes are reduced somewhat even from $t=9$ to $t=10$; then diminishing cosine functions must be combined in a slightly different way so as to add up to the same rectangle that is formed by cosine functions of constant amplitude.

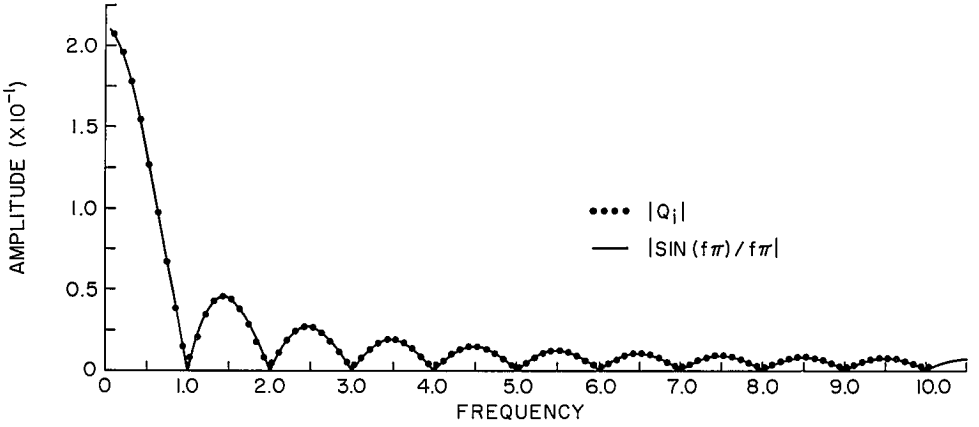


FIGURE 6. PLOT OF $| \sin(f\pi)/f\pi |$ AND $|Q_i|$ AT $t=9.5$ AT DIFFERENT FREQUENCIES

Recall that when deriving the spectrum of a rectangle of height H and width W that is repeated every T years, the phase angle $\theta_i = 0$. When representing the spectrum as $|\sin(x)|/x$ instead of $\sin(x)/x$, we need to modify the concept of θ slightly because $|\sin(x)|$ can never be negative, whereas $\sin(x)$ takes on both positive and negative values. We may think of the θ_i , when the spectrum is defined as a plot of amplitude vs. frequency, as determining the proper sign. Hence, at frequencies from 0 to $1/W$, $\theta = 0$; at frequencies from $1/W$ to $2/W$, $\theta = \pi$; at frequencies from $2/W$ to $3/W$, $\theta = 0$, etc. Hence, the plot of θ_i vs. frequency is simply a plot of flat lines alternating between values of 0 and π over intervals of width $1/W$. In Figure 7, this "theoretical" plot is compared with the actual plot of the actual phase angles θ_i in equation (17). The two are not identical, but the similarity between them is quite remarkable.

The pseudo-spectrum is of course gradually changing, as the components at different frequencies are subject to different rates of attenuation. At $t=19$, the midpoint of the triangular structure formed by the second generation births, all of the Q_i 's have been diminished by a factor of $e^{9.5x_i}$ from their value at $t=9.5$. Hence $|Q_i(9.5)| = |Q_i(0)| e^{9.5x_i}$; $|Q_i(19.0)| = |Q_i(0)| e^{19.0x_i}$;

$$\text{and} \quad \left| \frac{Q_i(19)}{Q_i(0)} \right| = \left| \frac{Q_i(9.5)}{Q_i(0)} \right|^2.$$

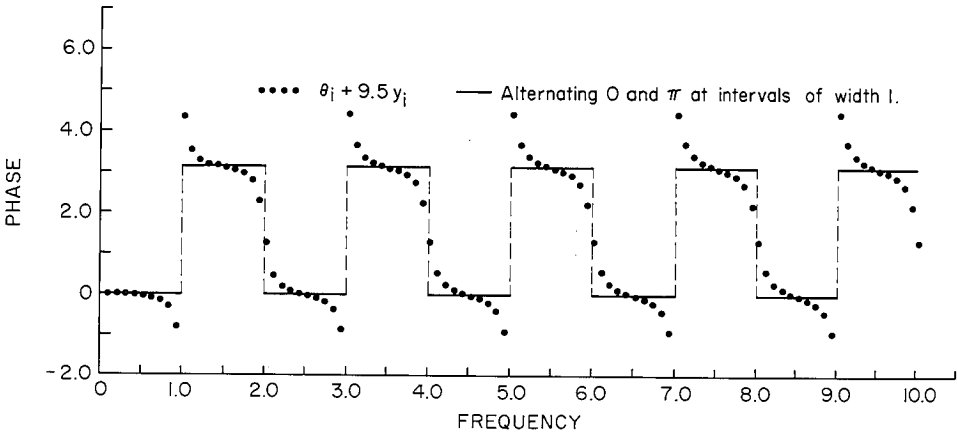


FIGURE 7. PLOT OF PHASE AT $t=9.5$ OF COMPONENT AT DIFFERENT FREQUENCIES

The spectrum of a population element at $t=0$ is "flat"; i.e., in the Fourier representative of a rectangle of infinitesimal width, $Q_i = Q_0$ at all frequencies, as can be seen from close examination of equation (16). Multiplying $Q_i(0)$ by $e^{9.5x_i}$ converts the flat spectrum of a population element into the $\sin(\pi f)/(\pi f)$ spectrum of an isolated rectangle one year wide. A second multiplication by $e^{9.5x_i}$ converts the spectrum into $(\sin(\pi f)/(\pi f))^2$ — precisely the Fourier spectrum of a repetitive sequence of births distributed in a two-year-wide triangular structure each generation — a structure that rises linearly to a peak in the first of two years, and declines linearly in the second.¹⁴ This structure is, of course, the structure of projected births in the second generation when $\phi(a)$ is a rectangle extending from age nine to ten. In other words, the Q_i 's calculated at $t=0$ approximate the Fourier spectrum of an element concentrated at a moment of time; the Q_i 's that exist at $t=9.5$ approximate the spectrum of a rectangle one year wide; and at 19.0 the Q_i 's form very nearly the Fourier spectrum of the triangular second generation births.

This birth sequence is almost periodic if rectangular $\phi(a)$ extends from nine to ten; but generation after generation the births alter — spreading out and becoming smoother in form: the reduction of amplitude at the higher frequencies corresponds to the spread of births over one more year in each generation, and to the progressively reduced jaggedness of the sequence. When, after about one hundred generations (nine hundred and fifty years), the first 44 per cent of generation $n+1$ births begins to precede the last 44 per cent of births of generation n , all but the lowest frequency variations have long since been reduced to insignificant levels. Indeed, by five hundred years the magnitude of all oscillatory terms except that with a 9.5-year cycle length is below 2.5 per cent of the average annual births. This term is still at 38 per cent of the average level, and does not fall below 2.5 per cent until more than two thousand years have passed.

Quickly converging birth sequences, with absolute convergence

At the other extreme from a rectangular $\phi(a)$ in which reproduction begins only a little short of its termination is a $\phi(a)$ in which reproduction begins at birth — a schedule of net reproduction that is a rectangle from age zero to age β . When $\alpha = .9\beta$, as in the example just analyzed, births in the successive generations that are the descendants of a given cohort do not overlap for nine generations, and do not overlap extensively for about one hundred generations. If reproduction begins at age zero, all generations overlap from the start — mother, daughter, and granddaughter can begin life at essentially the same moment. We saw at an earlier point that the births in the n^{th} generation are the n^{th} convolution of $\phi(a)$ with itself; in the preceding

example in which early generations did not overlap, this formulation determined the changing shape of the isolated islands of births from years 9 to 10, 18 to 20, 27 to 30, and so on. When rectangular fertility is concentrated near β the very slow progress to a stable sequence of births involves the gradual broadening of births in later generations until the overlapping of births from many generations produces a perfectly smooth sequence.

In the case we are now considering, the births of all successive generations overlap in the first interval from zero to β years. Convergence in this instance includes rather abrupt variations in the early part of the sequence $B(t)$ (the progeny, again, of a population element) as the births in the early generations having been confined to a short interval, rather abruptly withdraw.

For greater simplicity of calculation and notation, we shall let $\beta=1.0$ so that the net fertility function extends from zero to one, and the net reproduction rate is the height of the fertility function in this unit interval. Let us further suppose that the net reproduction rate is one, so that the stable birth sequence is a horizontal line. The birth sequence (if the initial population is a cohort concentrated at age zero) is the sum of $B_1(t)$, $B_2(t)$, $B_3(t)$ where $B_i(t)$ is the sequence of births in the i^{th} generation, and is the convolution of $\phi(a)$ with itself i times. But now the births in all generations begin at $t=0$. Thus $B(t)$ is the sum of all convolutions (up to an infinite order) of a uniform distribution of fertility from zero to one. The underlying relation between births in two successive generations is $B_n(t) = \int_0^1 B_{n-1}(t-a)da$. It follows that the rate of change of $B_n(t)$ is $\frac{dB_n(t)}{dt} = B_{n-1}(t) - B_{n-1}(t-1)$. The logical basis of this relation is simple: the rate of increase of $B_n(t)$ is enhanced by every additional current birth of a member of the preceding generation (subject immediately to a unit level of net fertility), and is diminished as each person born a year ago reaches the age (one) at which net fertility falls to zero. But for t less than one $B_{n-1}(t-1)$ is zero, hence up to one year $\frac{dB_n(t)}{dt} = B_{n-1}(t)$. It is obvious that the population element B_0 produces a constant stream of births during the first year. That is $B_1(t) = B_0$, for t less than one. Therefore $\frac{dB_2(t)}{dt} = B_0$, or $B_2(t) = B_0 t$ ($0 < t < 1$); and, over the same range $B_3(t) = B_0 t^2/2$, $B_4(t) = B_0 t^3/3!$, etc. The overall birth sequence from 0 to 1, is, then $B_0 \sum_{n=0}^{\infty} \frac{t^n}{n!}$, or $B_0 e^t$. (If the net reproduction rate were R instead of one, $B(t)$ from zero to one is $B_0 Re^{Rt}$.) For $t < \beta$, then, a population element subject to a uniform net fertility schedule with a net reproduction rate of R produces an exponential birth sequence (with an exponent of R/β). Note that the exponent is very different from the exponent of the stable birth sequence, which is r , equal to $(\log R)/T$, where T is the mean length of generation. In our example with $R=1.00$, and $\beta=1$, the birth sequence rises during the first year at a rate that multiplies births by e (2.718) every year; the long-run growth rate is zero. An initial exponential with a rapid increase has a simple explanation: if the uniform net fertility function persisted forever, it would be a special case (with $k=0$) of the net fertility function, $\phi(a) = m e^{-ka}$, that we showed earlier led to an immediate purely exponential birth sequence. It could be said that a cohort subject to a uniform $\phi(a)$ from birth does not necessarily know that reproduction is scheduled to stop at age one; until $t=1$ the sequence of births is the same (exponential) as if uniform fertility were going to continue forever.

At $t=1.0$, however, the births of each generation move to a new segment.¹⁵ First generation births abruptly cease, second generation births change from linearly rising to linearly declining, third generation births change from a second order polynomial with a positive second derivative to one with a negative second derivative.

With this net fertility function, indeed, an explicit expression for the birth sequence in each annual segment (not just for the first year) can be derived. For all births at any time t , the

relation $B(t) = \int_0^t B(t-a)\phi(a)da$ implies $\frac{dB(t)}{dt} = R(B(t) - B(t-1))$ or

$$\frac{dB(t)}{dt} - RB(t) = RB(t-1), \quad (18)$$

where R is the net reproduction rate. If $B(t)$ is known for one interval (0 to 1, for example), equation (18) provides the basis for calculating $B(t)$ in the next interval (1 to 2), and so on, by iteration. Suppose that in the interval $m-1$ to m , $B(t) = (r_0 + r_1 t + \dots r_{n-1} t^{n-1})e^{Rt}$. Assume that in the interval m to $m+1$, $B(t) = (s_0 + s_1 t + \dots s_n t^n)e^{Rt}$. Substitution in (18) leads to

$$(s_1 + 2s_2 t + \dots + n s_n t^{n-1}) = -\frac{R}{e^R} (r_0 + r_1(t-1) + \dots r_{n-1}(t-1)^{n-1}). \quad (19)$$

so that if each s_i (for $1 \leq i \leq n$) is assigned a value that satisfies equation (19), and a value of s_0 is found that produces a $B(t)$ at the lower boundary of the m^{th} interval properly matching the value at the upper boundary of the $(m+1)^{th}$ interval, $(s_0 + s_1 t + \dots s_n t^n)e^{Rt}$ is a correct expression for $B(t)$ in the $(m+1)^{th}$ interval. Solving (19) for $s_1, s_2 \dots s_n$ leads to:

$$\begin{aligned} s_1 &= M(r_0 - r_1 + r_2 + \dots + (-1)^{n-1} r_{n-1}) \\ s_2 &= \frac{M}{2} [r_1 - 2r_2 + \dots (-1)^{n-2} r_{n-1} (n-1)] \\ s_3 &= \frac{M}{3} [r_2 - 3r_3 + \dots (-1)^{n-1} r_{n-1} \left(\frac{(n-2)(n-1)}{2} \right)] \end{aligned} \quad (20)$$

where $M = -R/e^R$. The value of s_0 is determined from the value of $B(t)$ at $t = m$ (the lower boundary of the interval) which is obtained from the known value at the upper boundary of the earlier interval. $B(2), B(3), B(4)$, etc. can all be inferred directly from the upper bound of the next lower interval. However, at $t = 1$, where the original population element passes the upper limit of reproduction, there is a discontinuity: the beginning value of $B(t)$ in the second year is R less than the final value in the first year. The values of s_i in $B(t) = \sum_{i=0}^{m-1} (s_i t^i) e^{Rt}$ for the first four years are given in Table 1. Figure 8 shows the birth sequence for three years (3β years, the reader is reminded, since we have set $\beta=1$ to simplify calculations), as deviations from $B(0)e^{Rt}$, the ultimate stable sequence. Note that by year (3) the amplitude of variations is greatly reduced. The deviations from the exponential decrease very rapidly even when, along the exponential itself, births are doubling each year. Would this be true if fertility were very high, multiplying births each year by a ridiculously large number, such as 110,000?

Truly absolute convergence

Convergence to the stable population occurs because all fluctuating components of the birth series become a negligible fraction of the total; convergence is *absolute* if the magnitude of these components is a negative exponential function, and *relative* if the magnitude of any oscillatory component is constant or growing — in which case, of course, the rate of increase of the purely exponential component itself is always greater.

TABLE 1. VALUES OF THE COEFFICIENTS s_i IN $B(t) = \sum_{i=0}^{m-1} (s_i t^i) e^{Rt}$

m	R = 1				R = 2			
	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
1	1.0	--	--	--	2.000	--	--	--
2	1.0	-.3679	--	--	2.271	-.5413	--	--
3	1.0	-.5032	.0677	--	2.417	-.7611	0.07326	--
4	1.0	-.5779	.11746	-.0083	2.506	-.8801	0.12284	-.00661

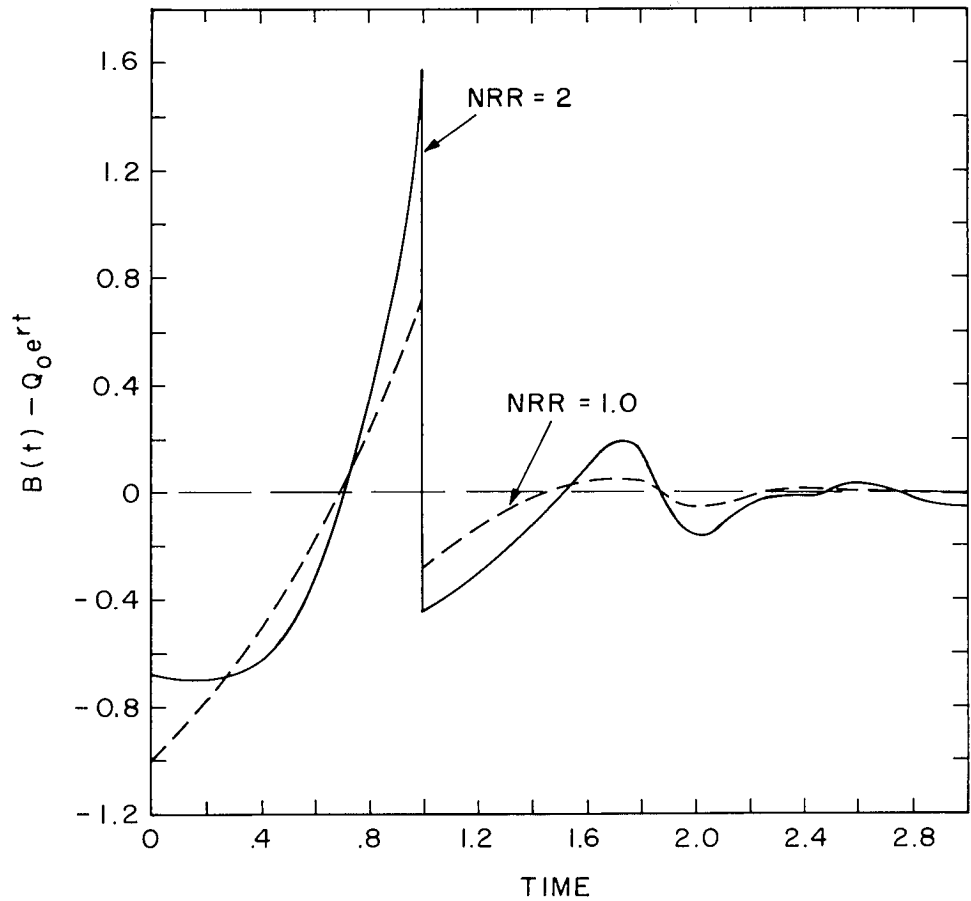


FIGURE 8. PLOT OF THE DIFFERENCE BETWEEN THE ACTUAL BIRTHS AND THE ULTIMATE STABLE BIRTH TRAJECTORY AGAINST TIME

Suppose $\phi(a)$ is modified by a small uniform proportionate increase in fertility at every age. The exponential rate of increase in the birth sequence is augmented slightly; the value of each y_i is slightly changed, and the value of each x_i is slightly increased. If the proportionate increase in $\phi(a)$ is very close to zero, the new values of x_i and y_i are very close to the old values; a plot of the values of x_i and y_i for a given $\phi(a)$ multiplied by continuously greater scalar multiples, would follow a continuous path. The natural expectation of the effect on x_i of higher fertility at every age is that x_i will be larger, just as r is larger. This kind of qualitative reasoning leads to the expectation that as the net reproduction rate is increased, for a given age structure of $\phi(a)$ the value of x_i will ultimately increase until a positive x_i is reached. In other words, it seems plausible that the absolute convergence that characterizes a birth sequence with a moderate rate of increase would become relative convergence if the level of fertility were sufficiently elevated. To the contrary, it can be shown that a positive value of x_i cannot occur when rectangular net fertility begins at age zero and cannot occur for y_i 's that fall in particular regions for a rectangular $\phi(a)$ in which α is equal to $\beta/3$ or less.

If oscillatory terms are nonincreasing no matter how rapid is the exponential increase in births, convergence is truly absolute. The proof of truly absolute convergence when $\phi(a)$ is a rectangle extending from zero to β follows from the nature of the sine function in:

$$R/\beta \int_0^\beta e^{-xa} \sin(ya) da = 0 \quad (21)$$

which is one of the two equations that x and y must satisfy. Suppose x were positive then; $e^{-xa} \sin(ya)$, a sine function that constantly diminishes in amplitude, would begin with a positive half-cycle larger than the ensuing negative half cycle; consequently the value of the integral at every point in the first cycle would be positive. Extension of the integral to additional cycles (or fractions thereof) would only add more net positive area. Thus, $R/\beta \int_0^\beta e^{-xa} \sin(ya) da$ can never equal zero when x is positive; it follows that x_i can never be positive when a rectangular $\phi(a)$ begins at zero, and that all oscillatory terms in $B(t) = Q_0 e^{rt} + \sum_{i=1}^{\infty} Q_i e^{r_i t} \cos(y_i t + \theta_i)$ must have negative values of x_i no matter how large R is. With trivial modifications this argument applies to any continuous nonincreasing $\phi(a)$ that begins at $a=0$.

If no positive value of x can occur in an oscillatory component when rectangular $\phi(a)$ begins at age zero, does the same truly absolute convergence hold for net fertility functions extending from s to 1 ($0 < s < 1$) if s is small but not zero? To answer this question we shall first determine for what values of s and y it is possible to have a function that satisfies

$$\begin{aligned} F \int_s^1 \cos(ya) da &= 1 \\ \int_s^1 \sin(ya) da &= 0 \end{aligned} \quad (22)$$

Values of s , F , and y that satisfy (22) indicated the existence of net fertility functions extending from s to 1, such that for some fertility level F (net reproduction rate $F(1-s)$) there is an oscillatory component at a frequency of $(y/2\pi)$ cycles per year such that $x = 0$. The values of s and y for which x can be zero define the boundary of the region in which a positive x is possible. When equation (22) is integrated, the result is

$$\begin{aligned} \sin(y) - \sin(ys) &= y/F \\ \cos(y) &= \cos(ys) \end{aligned} \quad (23)$$

The only values of y and ys that make the cosines equal and create a positive difference between the sines are values such that $y + ys = n \cdot 2\pi$ and such that y lies in the first or second

quadrants (i.e. $2r\pi < y < (2r+1)\pi$) so that $\sin(y) - \sin(ys) (=2\sin(y))$, when $y+ys = n \cdot 2\pi$ is positive. The two conditions that y and s must satisfy so that x can be zero are:

$$y = \frac{2n\pi}{1+s} \quad (24)$$

$$2r\pi < y < (2r+1)\pi$$

for any integers n and r . For each value of the positive integer r there are r different values of n such that (24) can be satisfied, within a calculable range of s . The values of y that allow x to be zero for specified ranges of s are shown in Table 2. The first range of y ($r = 1$) that permits for appropriate s , an x of zero corresponds to the range of y for the lowest frequency oscillatory term for each s , the second range to the second lowest frequency term, etc. This table is interpreted as follows:

- a) For $s > 1/3$, positive values of x_1 are possible, but not for $s \leq 1/3$.
- b) For $s > 3/5$, and $\frac{1}{5} < s < \frac{1}{2}$, positive values of x_2 are possible, but not for other values of s .

As the value of r increases (in effect the *order* of the oscillatory term) the minimum value of s for which x can equal zero is smaller. If s is some positive number ϵ , no matter how small, it is possible, with sufficiently large F , to have an oscillatory term in the birth sequence that grows larger; but if s is small, the only oscillations that can increase are at high frequency, and if $s = 0$, no oscillations of increasing magnitude are possible.

The net reproduction rate causing x to equal zero can be calculated from equation (23), from which we see that the constant level of fertility F , from s to 1, that yields an x of zero for a given y is $F = y/(\sin y - \sin ys)$; but, at values of y such that $\cos(y) = \cos(ys)$, $\sin(y) = -\sin(ys)$, or $F = y/2\sin(y)$. The net reproduction rate R is $F(1-s)$, or, for $x = 0$,

$$R = \frac{y(1-s)}{2\sin(y)}. \quad (25)$$

We can now calculate how large the net reproduction rate must be for $x=0$ given any value of s . For example, let $s = 0.1$; from the general form of values of r, n, y , and s given in the last panel of Table 2, we see that $s = 0.1$ lies between $\frac{1}{2p+1}$ and $\frac{1}{p}$ for $5 \leq p < 10$, and lies between $\frac{3}{2p+1}$ and $\frac{2}{p}$ for $15 \leq p < 20$; etc. Therefore, there are an infinite number of values of y , which from (24) equals $\frac{2(p+1)\pi}{1.1}$, such that x could equal zero. However, to find the *smallest* value of R that makes $x=0$, we wish to select a y , subject to the above constraints, that produces the smallest value of $\frac{y(1-s)}{2\sin(y)}$. We would expect that this value would result from a y associated with the

smallest range of values of p , since $R = \frac{1.8\pi(p+1)/1.1}{2\sin(2\pi(p+1)/1.1)}$. Trial calculations show this expectation to be true. The values of y , $\sin(y)$, and R are presented in Table 3 for the five values of p in the smallest range ($5 \leq p < 10$). Not surprisingly, the smallest R comes at the middle. When $s=0.1$ it would require a net reproduction rate of over 20 to produce an oscillatory component at a frequency of 7.3 cycles per generation that did not diminish with time, and all other oscillations would diminish at that net reproduction rate.

TABLE 2. SETS OF VALUES OF x AND y WHICH SATISFY EQUATIONS (24) FOR ALL POSSIBLE VALUES OF n AND r

r	n	value of y	range of s
1	2	$2\pi < y < 3\pi$	1/3 to 2/2
2	3	$4\pi < y < 5\pi$	1/5 to 2/4
	4		3/5 to 4/4
3	4	$6\pi < y < 7\pi$	1/7 to 2/6
	5		3/7 to 4/6
	6		5/7 to 6/6
p	$p + 1$	$2p\pi < y < (2p+1)\pi$	$1/(2p+1)$ to $2/2p$
	$p + 2$		$3/(2p+1)$ to $4/2p$
	.		
	.		
	.		
	$2p$		$(2p-1)/(2p+1)$ to 1.0

TABLE 3. VALUES OF ρ , y , $\sin(y)$ AND R SUCH THAT $x = 0$

Values of p such that $x = 0$	$y = \frac{2(p+1)\pi}{1+s}$	$\sin(y)$	$R = \frac{y(1-s)}{2 \sin(y)}$
5	10.91π	.2789	55.3
6	12.73π	.7501	24.0
7	14.55π	.9876	20.9
8	16.36π	.9048	25.6
9	18.18π	.5358	48.0

In contrast, if $s = 0.9$, the smallest value of R that makes $x = 0$ is associated with the lowest frequency oscillation, where $y = \frac{4\pi}{1.9} = 6.614$. For this y to make $x = 0$, $R = \frac{(6.614)(.1)}{2\sin(2.1053\pi)} = \frac{.6614}{.6494} = 1.018$. A growth of only 1.8 per cent per generation in births would be accompanied by oscillations that did not diminish.

Is a human fertility schedule possibly compatible with oscillations that increase? Could a human fertility schedule yield relative rather than absolute convergence? None actually recorded has this feature, so that the practical answer is negative. If, however, the rate of childbearing among married Hutterites at ages 20-25 were experienced in a population with no fertility outside that age range, and if mortality were that of contemporary America women, $\phi(a)$ would be essentially a rectangle from 20 to 25, equivalent, except for scale, to a rectangle from .8 to 1.0. To have oscillations with no damping, such a $\phi(a)$ would require a net reproduction rate of 1.086. Hutterite marital fertility from 20 to 25 in 1921 to 1930 was 0.55; with at least 97 per cent surviving to age 25, the NRR would be above 1.25, more than sufficient to produce growing oscillations.

Footnotes

1. More precisely $m(a)da$ is the proportion of women bearing a daughter in a very short age interval a to $a+da$.
2. See Coale [1972], Chapter 3 or Keyfitz [1968], Chapter 5.
3. For the mathematically inclined, we note that the birth sequence takes on this form because there is only one real (non-oscillatory) root of equation (3), that it is larger than the real component of any complex root, and the oscillatory terms become negligible relative to the one non-oscillatory exponential. The oscillatory terms themselves do not disappear unless all real parts of the complex roots are negative (a situation we refer to later as absolute convergence). However, the oscillatory terms do *always* become arbitrarily small relative to the real root.
4. The mathematical reader will note that the two modes are really two forms of exponential sequences, one with a real exponent and the other with a complex exponent.
5. This value is the sole real root of the equation $\int_a^\beta e^{-ra}\phi(a)da - 1 = 0$. The complex roots are equivalent to the product of an exponential and a sinusoidal component that appears in equation (8) below. It is the roots of this integral equation that govern the path to stability, hence the title of this article.
6. These are well-known trigonometric identities which are proved in any text book of trigonometry. They also follow very simply from the relation $e^{ix} = \cos(x) + i\sin(x)$, where $i = \sqrt{-1}$.
7. The value of Q^0 is given by:

$$Q_0 = \frac{\int_a^\beta F(t)e^{-rt}dt}{\int_a^\beta a\phi(a)e^{-ra}da}$$

$$\text{and } Q_i = 2\sqrt{\frac{M_i^2 + N_i^2}{G_i^2 + H_i^2}}$$

$$\text{and } \theta_i = -\arctan \frac{M_i H_i - N_i G_i}{G_i^2 + H_i^2}$$

when $F(t)$ is the births that occur to the initial populations, and

$$M_i = \int_0^\beta F(t)e^{-\lambda_i t} \cos(y_i t)dt$$

$$N_i = \int_0^\beta F(t)e^{-\lambda_i t} \sin(y_i t)dt$$

$$G_i = \int_0^\beta a\phi(a)e^{-\lambda_i a} \cos(y_i a)da$$

$$H_i = \int_0^\beta a\phi(a)e^{-\lambda_i a} \sin(y_i a)da$$

[adopted from formulas given in Lotka, (1939). See Coale (1972, pp. 67-69).]

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8. The equations which determine the parameters of the oscillatory components, $\int_0^T e^{-\lambda a} \cos(\gamma a) \phi(a) da = 0$, and $\int_0^T e^{-\lambda a} \sin(\gamma a) \phi(a) da = 0$, become, in this instance, $\int_0^T m e^{-(k+\lambda)a} \cos(\gamma a) da = 1$, and $\int_0^T m e^{-(k+\lambda)a} \sin(\gamma a) da = 0$. Integration leads to the solutions $y=0$; $x=m-k$, values of x and y which simply replicate the exponential.
9. Convergence is also avoided if $\phi(a)$ has positive values at several exact ages that have a common sub-multiple other than 1.0, but this more complicated example will not be discussed.
10. Note that one cannot simply evaluate the integral since the fertility rate would have to become arbitrarily large as the width of the interval shrinks to zero. Instead, imagine that the fertility distribution shrinks to a single age β . The probability of having R births at age β is 1.0. Then the integral equation is replaced by $R e^{-\lambda \beta} = 1.0$.
11. This relationship, of course, follows immediately from the fact that the mean of a sum of random variables is the sum of the means of each and the variance of a sum of *independently* distributed random variables is the sum of the individual variances.
12. Stated precisely, if the random variables X_1, \dots, X_n form a random sample of size n from a given distribution with mean μ and variance σ^2 , then for any fixed number x , $\lim_{n \rightarrow \infty} \Pr \left[\frac{\sum_{i=1}^n (X_i - \mu)}{\sigma} \leq x \right] = \Phi(x)$, where $\Phi(x)$ is the distribution function of the standard normal distribution.
13. Any function subject to certain restrictions relating to continuity can be so represented; these conditions are met by any of the birth sequences considered here.
14. The triangular structure is, like the rectangle, symmetrical about an origin at the midpoint of the second generation

births. Hence $Q_i = C_i = \frac{4}{T} \int_0^T (1-t) \cos\left(\frac{2i\pi}{T}t\right) dt$. Integrating leads to $C_i = \left(\frac{4}{T}\right) \left(\frac{T}{2i\pi}\right)^2 (1 - \cos\frac{2\pi i}{T})$;

but since $1 - \cos(x) = 2 \sin^2\left(\frac{x}{2}\right)$; $C_i = \left(\frac{2}{T}\right) \frac{\sin^2(i\pi/T)}{(i\pi/T)^2}$.

Since $Q_0 = \frac{2}{T}$, then $Q_i/Q_0 = [\sin(i\pi/T)/(i\pi/T)]^2$.

15. The n^{th} order convolution of a uniform distribution consists of n segments, each of duration equal to interval (α to β in general, zero to one in this instance). Adjacent segments share the same value of $B(t)$ and the first $(n-1)$ derivatives of $B(t)$ at the boundary, but the n^{th} derivative changes.

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Received October, 1980; Revised June, 1982.

OUT-MIGRATION, FERTILITY, MORTALITY AND NATURAL INCREASE DURING THE DEMOGRAPHIC TRANSITION

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Résumé— Cette étude présente un modèle multivarié qui esquisse 1) les voies casuelles qui se trouvent dans l'hypothèse qui admet que la migration externe ralentit les baisses du taux de natalité par suite de la pression démographique; 2) les relations de la migration externe et de l'augmentation naturelle avec les baisses en taux de mortalité rural; et 3) l'importance relative des changements en taux de natalité versus des changements en taux de mortalité dans la cause des changements en augmentation naturelle rurale. L'utilité par une analyse longitudinale des données démographiques de l'Angleterre et du Pays de Galles (1820-1930) et de la Suède (1750-1920).

Abstract— This paper presents a multivariate model which delineates: 1) the causal paths underlying the hypothesis that out-migration slows declines in the rural birth rate due to population pressure; 2) the relationships of out-migration and natural increase with declines in the rural death rate; and 3) the relative importance of changes in the birth rate versus changes in the death rate in causing changes in rural increase. The utility of the model is illustrated by a longitudinal analysis of demographic data from England and Wales (1820-1930) and Sweden (1750-1920).

Key Words— demographic transition, out-migration, fertility, mortality

Introduction

The hypothesis that out-migration represents a significant demographic response which may produce variations in the process of the demographic transition had been discussed in the transition theory literature for several years. Davis (1963) lists out-migration along with celibacy, sterilization, abortion and postponement of marriage in outlining the components of a "Multiphasic Response" to population pressure resulting from lower mortality. Friedlander (1969) presents the most extensive discussion of migration, *per se*, arguing that migration from rural areas slows the demographic transition by providing an escape valve for the population pressures that arise when mortality rates decline. Either emigration from a country or migration to urban areas which need manpower is thought to serve this function. It has become customary to reference Friedlander's work (Easterlin, 1976; Nam and Gustavus, 1976; Macisco, 1972; Mott, 1972; Uhlmann, 1972; Zelinsky, 1971a), but at present this interesting research line has not been thoroughly investigated.

The purpose of our study is to present a model which: 1) examines the causal paths underlying the out-migration hypothesis; 2) includes the relationships of out-migration and natural increase with declines in the death rate; and 3) specifies the relative importance of declines in the birth rate and declines in the death rate in causing changes in natural increase. Following Friedlander's analysis, this paper investigates the extent to which out-migration

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