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On the Leontief Structure of Household Populations

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Abstract

In a given population, we consider the age distribution of all persons in households and the age distribution of household-heads or household-markers. We show that the formal relationship holding between the two age distributions is equivalent to the input-output relationship in the Leontief model of the open economy. The notions of household composition and household accommodation which have emerged independently over the past two decades, are shown to be formally linked within this relationship.

Résumé

Dans une population donnée, nous considérons la structure par âge de toutes les personnes des ménages et celle des chefs ou soutiens de ménage. Nous montrons que le rapport formel existant entre les deux structures d'âge équivaut au rapport apports-productions de la méthode due à l'économiste Leontief. Il s'avère qu'il existe un lien formel – au sein de ce rapport – entre les principes de composition et de logement des ménages qui se sont manifestés indépendamment au cours des vingt dernières années.

Key words: households, age-distribution, Leontief, input-output

Introduction

The formal linkage between distributions of household-persons and household heads has been the subject of only a modest analytic research in household demography (for a review, see De Vos and Palloni, 1990; Keilman and Keyfitz, 1988: 272-285). In the late 1980s Pitkin and Masnick introduced the notion of *household accommodation*, providing one aspect of such formal linkage in a ratio between household-persons who are 'non-heads', and all household persons. Pitkin and Masnick define an *age-specific* accommodation rate as the total number of persons in households whose heads are in the *j*th age group, divided by the total number of persons in the *j*th age group. Age-specific accommodation rates can be also perceived as the column sums in an ordered matrix of elements referred to as t_{ij} . Each element t_{ij} indicates the ratio of non-heads in the *i*th age group affiliated with household heads in the *j*th age group, to the total number of persons in the *j*th age group. The ordering of elements t_{ij}, in fact, yields a square matrix that may be assumed, in general, to be nonsingular (Pitkin and Masnick, 1987: 317-318).

We consider here the average number a_{ij} of persons in the *i*th age group *per* household whose head is in the *j*th age group. These ratios, too, have been shown ordered in a square matrix (Akkerman, 1980). More recently, Murphy (1991: 168-169) had suggested that the ratios t_{ij} and a_{ij} might be related. In the following we show that the ratios t_{ij} and a_{ij} are, indeed, analytically related, and that this relationship is formally equivalent to a relationship holding in the open Leontief model in economics. An application to the same data that were used by Pitkin and Masnick will illustrate the analytic relationship between the ratios t_{ij} and a_{ij}.

Following a suggestion by William Brass (1983), attempts have been made to avoid the reference to household-heads, and to use the term 'household-markers' instead (e.g., Murphy 1991: 158-159). In the traditional demographic setting, however, both household composition and household accommodation refer to household-*heads* rather than to household-markers. For consistency with accepted notation in contemporary demographic statistics, and to the extent possible, the generic term household-*marker* will be used here to identify *one* person who is considered to be the reference person to the household of his or her affiliation. In order to avoid awkward terminology we shall retain the term 'headship' as indicating the attribute of a household-person to be a household-marker.

Household Populations

A household population is a population of persons within all households. Each household has one and only one household-marker, and thus all household-markers form a subset within the population of household persons. We assign the term `household member' to any household-person who is not a household marker. Accordingly, a household person is either a household marker or a household member. We also accept that each household person belongs to one and only one household, and that each household includes one or more household-persons. For the sake of brevity, we refer on occasion to household-persons as *persons*, to household-markers as *markers*, and to household-members as *members*.

We consider the population to comprise n age groups, although the following considerations are applicable to any mutually exclusive population groups of the same domain. The number of persons in the ith age group is w_i, and the distribution of persons is given in a column vector w, $w' = (w_1, w_2, ..., w_n)$. The number of household-markers in the ith age group is k_i, and the distribution of household-markers is given in a column vector k, $k' = (k_1, k_2, ..., k_n)$.

Consider all those persons in an age group who are affiliated with household markers in the same, or in another, age group. We introduce now the value a_{ij} as the ratio of *all* persons in the ith age group who are affiliated with household markers in the jth age group, to the number of household markers in the jth age group. The value a_{ij}, then, is the average number of persons in the ith age group *per* household whose household-marker is in the jth age group ($i, j = 1, ..., n$). This is the essence of the numeric notion of household composition, linking the average number of persons in a given age-group to their respective *one* household-marker who is in the same or any other age-group. The notion of household composition has been shown useful, in particular, in the modelling of household and population change over time (Akkerman, 1996).

Consistent with the conventional measurement of population, the demographic assessment of households has often attained a temporal perspective. Average household size, for example, has been mostly seen as the result of factors operating *over time*, such as mortality, fertility, rules of residence and age at marriage (Burch, 1970). The usefulness of temporal measurement is unequivocal, mainly due to its implication for forecasting; yet, the structure of household composition, *at a single point in time*, deserves attention in its own right.

Household Composition

The formal relationship between the distribution of household persons and the distribution of household markers has been shown to be (Akkerman, 1980):

$$w = A\,k, \qquad (1)$$

A being a square matrix, $n \times n$, of ordered elements, $(a_{ij} + \delta_{ij})$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, ..., n)$. Each element $(a_{ij} + \delta_{ij})$ of A denotes the average number of persons in the ith age group per household whose marker is in the jth age group. The matrix A, referred to as the *household composition matrix*, has been assumed to be nonsingular, thus allowing for the reciprocal relationship,

$$k = A^{-1} w. \qquad (2)$$

When household markers are considered to be household *heads*, the assumption has been that the number of household heads in the first age group is 0, or $k_1 = 0$. Accordingly, it has been customary to partition the matrix A, and write it as

$$A = \begin{bmatrix} 1 & a \\ 0 & C \end{bmatrix}$$

where the number 1 is the value of the element a_{11} set by convention, a is a row vector of dimension $(n\text{-}1)$, and C is a nonsingular matrix $(n\text{-}1) \times ((n\text{-}1)$.

Household composition, thus, is a demographic structure that expresses, in a set of ratios, the affiliation of household-persons with their respective household-markers. It has been noted recently that the demographic structure described by Equation (1), resembles the formal pattern of the Leontief Input-Output model of the open economy (Akkerman, 1996).

We show in the following that the demographic structure of population and households, at a single point in time, and the formal structure of the open Leontief model, are in fact equivalent. Specifically, we demonstrate that the relationship between the population distribution, and the distribution of household-markers (usually, household-heads), at a single point in time, has formal attributes identical to those holding in the Input-Output model of the open economy.

As evident from (1) and (2), the household composition matrix A is instrumental in the relationship between household and population distributions. We show here that the relationship facilitated by the matrix A at a *single point in time* is equivalent to a relationship expressed in the open Leontief model, over a single *interval* of time. The demographic system of a population of household-persons whose age distribution is linked to the age distribution of household-markers by the matrix A will be referred to as the Leontief structure of a household population.

Household Accommodation

The Leontief structure of the household population emerges from a relationship holding between the formal notions of household composition and household accommodation. Pitkin and Masnick (1987: 318) consider the number m_{ij} of all household-members in the ith age group, affiliated with household-markers in the jth age group. They show the ratio,

$$t_{ij} = m_{ij}/w_j$$

as the *proportion* of members m_{ij} in the ith age group affiliated with markers in the jth age group, within the total number w_{ij} of persons in the jth age group. The sum of elements t_{ij} in each column j has been referred to as the *age-specific accommodation rate* (Pitkin and Masnick 1987: 317).

Consider now the number of persons in the ith age group. Some persons are household-members, and some others are household-markers. In fact, from the definition of t_{ij} it follows that the number of household-members in the ith age group, in the whole population, is

$$\sum_{j=1}^{n} t_{ij}\, w_j,$$

so that the number of household-*markers* in the ith age group is,

$$k_i = w_i - \sum_{j=1}^{n} t_{ij}\, w_j. \qquad (3)$$

Following Pitkin and Masnick we further consider a square matrix T, $n \times n$, of the ordered values t_{ij} $(i, j = 1, ..., n)$. The relationship (3) can be now rewritten for all n age groups in a standard matrix notation as,

$$k = w - T\,w,$$

or

$$k = (I - T)\,w, \qquad (4)$$

where I is the identity matrix. Equation (4) is precisely the form of the open Leontief model.

Input-output and household composition

The open Leontief model (4) is a mechanism that *forecasts* a distribution w of all goods produced, as *gross output*, at the end of a unit time interval, from the distribution k of goods presented by outside demand at the beginning of the time interval. The forecast is due to the definition of m_{ij}, in the Leontief model, as the volume of goods flowing over a unit time interval from sector i of the economy to sector j. Accordingly, t_{ij} in the Leontief model is the number (or the fraction) of units of the ith good at the beginning of the interval that result in the production of one unit of the jth good by the end of the interval.

Observing Equation (4), of particular interest is the case where $(I - T)$ is a nonsingular matrix. In fact, when T is nonsingular, a common occurrence in a matrix of the type $(I - T)$ is that it too is nonsingular and that its inverse, $(I - T)^{-1}$, is a nonnegative matrix (Berman and Plemmons, 1979: 132-133). The matrix $(I - T)^{-1} \geq 0$, within the context of the original Leontief model, has been on occasion referred to as the Leontief inverse. From (2) and (4) it can be conjectured immediately, that a nonsingular household composition matrix A is equivalent to the Leontief inverse:

$$A = (I - T)^{-1} \qquad (5)$$

Conversely, the matrix A^{-1}, viewed as the household composition inverse,yields:

$$A^{-1} = (I - T) \qquad (5^*)$$

The mathematical structure (4) is often associated with formalization involving many problems in biological, physical and social sciences (Berman and Plemmons, 1979: 132-133, 242-245). The algebraic equivalence between household populations, as expressed in Equation (4), and the open Leontief model appears to be another illustration of this formalism. Nevertheless, a fundamental difference between the original Leontief model and the demographic system presented in (1) and (4) must be noted: The major substantive variation between the original economic function of the Leontief model, as opposed to the demographic system considered here, is the temporal context of the two.

The temporal meaning of Equation (4) within the demographic system considered here is fundamentally different from the Leontief model. The values m_{ij} and the ratios a_{ij} and t_{ij} all refer to a relationship occurring at a single point in time, rather than over a time interval. The capability of the matrix A to project, over one or more time intervals, the distributions k and w has been explored elsewhere (Akkerman, 1996). The temporal mechanism within which A is utilized to project the household distribution k and the age distribution w, however, is different from (4).

Table 1. Household Accommodation, Submatrix Q, United States: 1970

Age of Member	Age of Household Head							
	20-24	25-29	30-34	35-44	45-54	55-64	65-74	75+
20-24	0.192	0.029	0.009	0.056	0.112	0.053	0.013	0.008
25-29	0.137	0.204	0.027	0.009	0.031	0.031	0.011	0.005
30-34	0.022	0.147	0.183	0.017	0.010	0.020	0.012	0.004
35-44	0.009	0.045	0.204	0.267	0.030	0.023	0.034	0.022
45-54	0.005	0.010	0.020	0.134	0.257	0.037	0.026	0.043
55-64	0.003	0.007	0.011	0.020	0.125	0.226	0.040	0.038
65-74	0.001	0.004	0.008	0.015	0.025	0.103	0.178	0.039
75 +	0.001	0.001	0.003	0.009	0.024	0.032	0.077	0.122
Total[a]	0.369	0.446	0.465	0.528	0.613	0.525	0.392	0.281

[a] Each Total in column *j* is the accommodation rate for members 20+ per household whose heads are in the *jth* age group.

Source: Pitkin and Masnick (1987, p. 318).

The Semantics of Household Composition

The values a_{ij} could be called *household composition ratios*. In order to examine closer the relationship between the matrices A and T it is useful to consider the sums of their respective elements in columns. The household composition ratios a_{ij} are by definition,

$$a_{ij} = m_{ij} / k_j \text{ for } i, j = 1, ..., n.$$

It has been pointed out that the sum of elements $(a_{ij} + \delta_{ij})$ in a column j yields the average size s_j of households whose markers are in the jth age group (Akkerman, 1996):

$$s_j = \sum_{i=1}^{n} (a_{ij} + \delta_{ij})$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, ..., n)$.

Average household size in the household population, then, is

$$(\sum_{j=1}^{n} k_j s_j) / \sum_{j=1}^{n} k_j.$$

Finally, we note that the reciprocal value of average household size yields the headship ratio (Murphy 1991: 158). The headship ratio, h, accordingly, is

$$h = \sum_{j=1}^{n} k_j / (\sum_{j=1}^{n} k_j s_j),$$

which conforms to the expression

$$h = \sum_{j=1}^{n} k_j / \sum_{j=1}^{n} w_j.$$

The household composition matrix A, thus, could be seen as an elaborate age-specific reference to the headship ratio. The meaning of the matrix T, and the ratios t_{ij}, is of interest here as well, due to the relationship that has emerged between the matrices A and T from (5). In the context of the Leontief model, the

Table 2. Submatrix ($I - Q$), United States: 1970

Age of Member	Age of Household Head							
	20-24	25-29	30-34	35-44	45-54	55-64	65-74	75+
20-24	0.808	-0.029	-0.009	-0.056	-0.112	-0.053	-0.013	-0.008
25-29	-0.137	0.796	-0.027	-0.009	-0.031	-0.031	-0.011	-0.005
30-34	-0.022	-0.147	0.817	-0.017	-0.010	-0.020	-0.012	-0.004
35-44	-0.009	-0.045	-0.204	0.733	-0.030	-0.023	-0.034	-0.022
45-54	-0.005	-0.010	-0.020	-0.134	0.743	-0.037	-0.026	-0.043
55-64	-0.003	-0.007	-0.011	-0.020	-0.125	0.774	-0.040	-0.038
65-74	-0.001	-0.004	-0.008	-0.015	-0.025	-0.103	0.822	-0.039
75 +	-0.001	-0.001	-0.003	-0.009	-0.024	-0.032	-0.077	0.878

Source: Table 1.

matrix T is usually referred to as the *input matrix*. A brief reflection shows that the matrix T, implicit in the demographic system (1), is a representation of the age-specific complement to the headship ratio.

In the demographic system (1) the generic value t_{ij} could be seen as the *complement* of the age-specific headship ratio h_j. The age-specific headship ratio is given as the proportion of household-markers in the jth group within the population in the jth group:

$$h_j = k_j / w_j. \qquad (6)$$

Analogously, the age-specific accommodation rate is (Pitkin and Masnick, 1987: 316-317; Murphy 1991, 108-109),

$$(w_j - k_j) / w_j. \qquad (7)$$

The expression in (7) is nothing but the sum of elements t_{ij} in column j ($j = 1, ..., n$). The complementarity of the age-specific headship ratio h_j in (6) with the age-specific accommodation rate in (7) is immediate.

The Leontief Inverse and the Household Composition Matrix

The household composition inverse, $A^{-1} = (I - T)$, is analogous to what has been known in the Leontief model as the *technology* matrix, and it belongs within the class of so-called M-matrices. Both the Leontief model and M-matrices have been the subject of extensive formal investigations. Appropriately, the matrix representation of household composition implies a significant potential for the further formal study of households. It is important, therefore, to also point out the substantively unique features in the demographic system presented here.

Within a temporal context, the demographic system (1) requires that there are no household-markers in the youngest age group (Akkerman, 1980). When household-markers are actually household-*heads* in the traditional sense, this formal condition is rooted in an obvious observation that headship does not commence prior to the age 15 or 20. In compliance with this observation a square submatrix C of A, as in the original relationships (1) and (2), consists only of those elements a_{ij} which refer to age groups that include household-markers, usually commencing with the age group 15-19 (Akkerman, 1996) or 20-24 (Pitkin and Masnick 1987: 318). Under the conditions of the demographic system (1) it can be shown easily that if relationship (5) exists for the matrices A and T, it also exists for the submatrix C and a corresponding submatrix Q of T, where both C and Q are nonsingular matrices of compatible dimensions. These relationships are succinctly illustrated by Tables 1 and 3.

Table 1 is a submatrix Q of the matrix T, reproduced directly from information by Pitkin and Masnick (1987: 318), which excludes data on the age-group 0-19. Each element t_{ij} of Q is as deliberated earlier, leading to the relationship (3); accordingly, sums of elements t_{ij} in the columns of Q are age-specific accommodation rates for population 20+.

Table 2 is the matrix $(I\text{-}Q)$ which, in turn, is a submatrix of $(I\text{-}T)$. Table 3 is the calculated inverse $(I\text{-}Q)^{-1}$ or, by definition, the submatrix C of the household composition matrix A. Each element of the calculated Table 3, is then an element a_{ij}, the average number of persons in the ith age group per household-head in the jth age group, $i, j = 2, \ldots, n$. The first age group, 0-19, has been omitted from C in correspondence to its omission from Q. The sums of elements a_{ij} in each column j of C, accordingly, yield the average number of persons age 20+ per household whose head is in the jth age group. The partial household composition matrix presented in Table 3 is therefore a direct estimation result from the household accommodation information provided by Pitkin and Masnick.

Conclusion

Until now, the numeric notion of household composition has been applied mainly within the context of modelling of household and population *growth*. In contradistinction to demographic growth over an interval of time, the present analysis concentrated on relationships holding at a single point in time. The significance of demographic analysis of household composition at a single point in time has the advantage of being applicable not only to age-distributions but to any other consistent stratification of population. The consideration of household composition at a single point in time can be broadened, therefore, to include, for example, occupation groups (Pitkin and Masnick, 1987: 318), population groups identified by place of daytime and night-time location (Akkerman, 1995), or age groups with intervals of unequal length, as the application here suggests.

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Table 3. Estimated Household Composition, Submatrix $C = (I\text{-}Q)^{-1}$, United States: 1970

Age of Person	Age of Household Head							
	20-24	25-29	30-34	35-44	45-54	55-64	65-74	75+
20-24	1.254	0.069	0.059	0.143	0.220	0.112	0.043	0.033
25-29	0.221	1.281	0.063	0.056	0.105	0.080	0.033	0.021
30-34	0.075	0.237	1.247	0.049	0.051	0.055	0.030	0.015
35-44	0.052	0.150	0.359	1.400	0.093	0.077	0.077	0.049
45-54	0.024	0.054	0.104	0.263	1.384	0.093	0.070	0.083
55-64	0.014	0.029	0.047	0.084	0.235	1.323	0.084	0.075
65-74	0.007	0.017	0.029	0.046	0.077	0.174	1.236	0.068
75 +	0.004	0.008	0.015	0.029	0.055	0.067	0.114	1.151
Total[a]	1.651	1.844	1.923	2.070	2.220	1.983	1.688	1.495

[a] Each Total in column j is the average number of persons 20+ per household whose head is in the jth age group.

Source: Table 2.

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