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A NOTE ON THE COMPONENTS OF COALE'S I_g AND OTHER INDIRECTLY STANDARDIZED INDICES

Thomas K. Burch and **Ashok K. Madan**
Population Studies Center, University of Western Ontario, London, Ontario, Canada

Résumé – Une note sur les composants de l'indice I_g de Coale et des autres indices calculés par le méthode des taux types. Quelques résultats de l'algèbre linéaire servent à éclaircir certains indices comparatifs. En utilisant la norme d'un vecteur de taux comme indice sythètique d'une série de taux démographiques, la relation $u \cdot v = |u|\ |v| \cos \theta$ (où $|u|$ et $|v|$ sont les magnitudes des deux vecteurs u et v, et θ l'angle entre eux) sert à donner une expréssion de l'intéraction entre les taux et la structure dans les indices comparatifs. Avec ces résultats et le modèle Coale-Trussell de fécondité légitime, nous indiquons que l'I_g de Coale se divise en trois parties: *a)* la magnitude, *b)* la répartition par âge, et *c)* une partie d'erreur qui résulte du choix arbitraire de la fécondité Hutterite comme point de repère. Nous proposons quelques corrections possibles aux indices comparatifs.

Abstract – Some results of elementary linear algebra are used to explore conventional standardized measures. Taking the norm of a rate vector as a summary measure of a set of demographic rates, the relationship $u \cdot v = |u|\ |v| \cos \theta$ (where $|u|$ and $|v|$ are the magnitudes of the two vectors u and v and θ the angle between them) is used to provide an expression for the interaction between rates and structure in standardized measures. Using these results and the Coale-Trussell model of marital fertility, it is shown that Coale's I_g can be factored into three terms measuring *a)* the scale, *b)* the pattern of marital fertility and *c)* an error term due to the arbitrary choice of Hutterite fertility as standard. Some suggestions are made in passing of possible corrections of conventional standardized indices.

Key Words – **standardization, Coale's I_g, interaction**

Thomas K. Burch and Ashok K. Madan

Introduction

An article by Roy and Nair (1983), on the decomposition of fertility indices, serves as a useful reminder that many apparently simple demographic measures are not unidimensional but comprise two or more distinct dimensions or components. In some cases, all the components are of substantive interest but confounded in a single index. Examples would be the scale ("volume") and shape ("pattern") components of fertility discussed by Roy and Nair and the "cross-term" or interaction component in comparisons across time or space by means of ratios or differences between rates. In other cases, some of the components reflect noise or error and are of no substantive interest. Prime examples are standardized rates or indices, which contain interaction terms reflecting the association between category-specific rates and population structure. These terms are erroneous, to the extent that they result from an arbitrary choice of standard rates or standard structure (indirect and direct standardization respectively). The use of different standards yields different interaction terms and thus different values of the resulting rate or index.

Roy and Nair (1983) deal with all the above problems in the context of Coale's index of marital fertility, I_g. The purpose of this note is twofold – to suggest a new approach to the study of level, shape and interaction in demographic measures, using results from vector alebra and to discuss Coale's I_g, using this alternate approach.

The techniques suggested are quite general, dealing with the central problem of conventionally standardized measures in any substantive area, namely the interaction resulting from a more or less arbitrary choice of standards. In place of traditional warnings about the dangers of comparisons among standardized measures (for example, Shryock and Siegel, II, 1973:419; Wunsch and Termote, 1978:54-55 and 58), this approach offers the hope of more specific guidance for diagnosis and possible correction of errors, although much more analytic and empirical work is still required.

Some Vector Algebra for Demographic Measures

Consider a set of demographic rates, u_i, where i refers to age category. u_i might represent, for example, a set of age-specific fertility rates for married women between the ages of 15 and 50, in five-year age groups. If we view this set of rates as a vector u in 7-space, then a natural summary measure of marital fertility is the norm or magnitude of the rate vector

$$|u| = \sqrt{u_1^2 + u_2^2 + \ldots + u_7^2} \quad (1)$$

In statistical terms, this is a measure of Euclidean distance (as distinguished from other possible distance measures) in n-space from the origin to the rate vector.

The conventional summary measure of u would be a sum of the individual u_i, such as the marital total fertility rate. What is the relationship between the norm of a vector and more conventional summary measures such as sums and averages? Mathematically, the sum and the norm are two out of a theoretically infinite number of distance functions on u. Gandolfo (1971:372) calls a function such as the total fertility rate (a sum) *absolute value distance* and $|u|$ (the magnitude of the rate vector) *Euclidean distance*. Of the latter he comments, "...this is perhaps the first function which everyone immediately thinks of, since it measures the length of the straight line segment joining the point with the origin...." An absolute value distance measure, by contrast, gives the distance to u traversed by moving in the direction of each successive axis to u_1, then to u_2,.... In 2-space, this can easily be visualized as the difference between moving to point (x,y) along the hypotenus of the right triangle formed by the vector and its projections on the x and y axes, versus moving first to x along the x axis and then to y, parallel to the y axis.

It can easily be shown that

$$|u| = \sqrt{n\,(\bar{u}^2 + s_u^2)} \quad (2)$$

where $\bar{u}$ and s_u are the mean and variance of the frequency distribution of rates (the distribution referred to here is not the more familiar age curve of fertility). The norm of a vector is a function of both the mean and variance of the frequency distribution of its elements, and of n, the number of elements. In demographic applications (for example comparing sets of marital age-specific birth rates), n will usually be constant (say n=7), and therefore ignored.

Thus, the norm of a vector of demographic rates, although a perfectly natural summary measure from the standpoint of linear algebra, is a strange hybrid when viewed in conventional statistical terms, and appears never to have been used in demographic analysis. The argument here is that it is one among several valid summary measures of fertility (just as the median, the arithmetic mean and the geometric mean are all valid measures of central tendency), and that its introduction might prove fruitful for certain analytic purposes (cf. Schoen, 1970).

In the present context, it is interesting to consider the empirical relationship between the norm and more conventional demographic measures. In the case of age-specific fertility rates, for example, for an international series of 73 sets of rates around 1960 (U.N., 1965), the correlation between the norm and the total fertility rate is +0.996. This high correlation is due, in turn, to the high correlation between the mean and variance as defined in Equation 2; in the series just mentioned, $r = +0.91$. Practically speaking, the norm of the fertility rate vector can serve as a proxy for the total fertility rate. A similar result may hold for many other types of rates, for example, mortality and nuptiality; this is a topic for further empirical work. The general condition for the high correlation between $\bar{u}$ and s_u is that sets of rates have similar shapes at different levels, or more formally that the different sets of rates be nearly proportional to one another (Yule, 1934).

With the norm of a vector thus established as a summary measure of demographic rates, we can recognize many demographic measures as dot (or scalar) products of a rate vector and a population vector, $u \cdot p$, and make use of the following relationship

$$u \cdot p = |u| \ |p| \ cos \ \theta \tag{3}$$

where θ refers to the angle between the two vectors (Strang, 1980:105ff). In other words, any demographic measure that can be expressed as a dot product can be factored into three terms, representing the magnitudes of two vectors and the angle between them.

Cos θ may be interpreted as a measure of the interaction or association between the two vectors (if the two vectors are expressed in deviation form, e.g., $u_i - \bar{u}$, then cos $\theta = r$, the familiar correlation coefficient). Cos θ = 1 when the angle is 0°, that is when the two vectors have the same direction. Another way of stating this is, for two vectors u and p, $u = kp$, that is the vectors are proportional to one another. In this situation, the largest rates (elements in u) are multiplied by the largest population numbers (elements in p), and the dot product $u \cdot p$ tends to be large. As θ increases, cos θ decreases, reaching 0 at $\theta = 90°$, when the vectors are orthogonal. Practically speaking, this situation will not occur in the present context, since it would require zero or negative elements in the vectors. It will be approached to the extent that the elements in the vectors are negatively correlated, so that large elements in one are multiplied by small elements in the other, resulting in a smaller dot product $u \cdot p$, even though $|u|$ and $|p|$ remain the same.

This is another way of expressing the familiar demographic point that, given two populations with equal total fertility rates and equal numbers of women in the reproductive ages, the total number of births can differ, depending on the interaction between the age structure of fertility and of population (Coale and Zelnik, 1963). The number of births in a population is the dot product of a vector of female age-specific birth rates and a vector of the female population by age, say $m \cdot f = |m|\ |f| \cos\theta$. It is thus the product of the level of fertility, female population size, and the interaction between the age patterns of fertility and of the female population. In this case, note that the interaction term, $\cos\theta$, is given by the data; that is, it operates in the real world.

In standardized measures, by contrast, the interaction results from an arbitrary choice of standard. Let m be actual age-specific fertility, f^* be a female population vector in a standard population, and P^* the total population. Then, a directly standardized birth rate

$$\frac{m \cdot f^*}{p^*} = \frac{|m|\ |f^*|\ Cos\ \varphi}{p^*} \tag{4}$$

contains an interaction term $\cos\varphi$ (where φ is the angle between the actual rate vector and the standard population vector) subject to the arbitrary choice of standard population by the investigator, a term that does not operate in the real world.

Similarly, in traditional indirect standardization, let m^* be a standard set of age-specific birth rates. Then a ratio of actual to expected births (I) has the form

$$I = \frac{m \cdot f}{m^* \cdot \mathrm{f}} = \frac{|m|\ |f|\ Cos\ \theta}{|m^*|\ |f|\ Cos\ \theta^*} = \frac{|m|\ cos\ \theta}{|m^*|\ cos\theta^*} \tag{5}$$

That is, it equals the ratio of the magnitudes of the actual and standard fertility vectors times the ratio of interactions between actual rates and structure and standard rates and structure. The latter ratio is in effect an error term, based on an arbitrary choice of standard rates.

Interestingly, even an equally weighted sum such as the total fertility rate contains an implicit interaction term.

$$TFR = 5 \Sigma m_i = m \cdot w = |m| \ |w| \ cos \ \theta$$

where $w = (5,5,\ldots 5)$. It involves a choice of weights that is arbitrary with respect to the real world. In vector terms, however, such equally weighted sums or averages seem natural enough, in that they are closely related to the orthogonal projection of the actual rate vector on the "diagonal" in n-space, considered as a neutral reference vector. It is neutral in the important sense that even radical changes in the age structure of the rates (e.g., reversing the order of two or more elements in m) leaves the resulting index unchanged (cos θ is constant for a given $|m|$). Weighted sums are natural in another sense as finite approximations of the area under the age curve (in this case of fertility) ${}_0\int^{\omega} m(a) \ da$.

Terminological Aside

Some clarifying comments are in order regarding the terms level, volume, scale, pattern, and shape when applied to fertility (or other demographic) rates. It is important to distinguish the frequency distribution of rates (e.g., the number or proportion of rates between 0.000 and 0.100, 0.100 and 0.200) and the age curve of fertility, or fertility rates as a function of age. Both involve "level" and "pattern" of fertility, but not in the same senses of those terms.

The frequency distribution of fertility rates can be described in terms of the mean (or closely related measures such as the total fertility rate), which measures level, and by higher moments, which reflect shape or pattern. As noted earlier (Equation 2), the norm of a rate vector as a summary measure of fertility reflects level and shape, insofar as it is a function of both the mean and the variance of the frequency distribution of rates. But this distribution contains no information on age, so that summary measures, such as a sum, average or norm of the rate vector, do not directly reflect the age pattern of fertility. Such measures are age-invariant, in the sense that their values would be unchanged if the age-order of the rates were reversed or otherwise radically altered, so long as the values of the individual rates remained the same.

The age-curve of fertility has shape or pattern in a different sense. This age pattern is given explicitly by the curve itself (for example, the ordered set of age-specific fertility rates) or by a functional representation of the curve. This age pattern is reflected in summary measures such as the norm or sum of rates only indirectly and implicitly, in the sense that the level of fertility can be thought of as resulting from both scale and shape. That is, given a reference set of rates, any other set of rates may be thought of as resulting from a twofold transformation of the reference vector, namely: *a*) a dilation or contraction (multiplication by a factor *k*) which changes scale, and *b*) a partial rotation (multiplication of individual rates by varying factors), which changes shape. More concretely, if we wish to lower fertility from that given by a set of age-specific rates, we can reduce rates "across the board" and/or, "bend the curve downward", at, say the upper ages. The Coale-Trussell model of marital fertility (to be discussed later) approaches things from this perspective. Figure 1 gives a graphic illustration. The top curve is a reference curve; the middle curve is that which results from multiplication of the reference by a constant scale factor; and the bottom curve, which has the smallest area or total fertility, results from multiplication by the scale factor and a varying (by age) factor relating to shape.

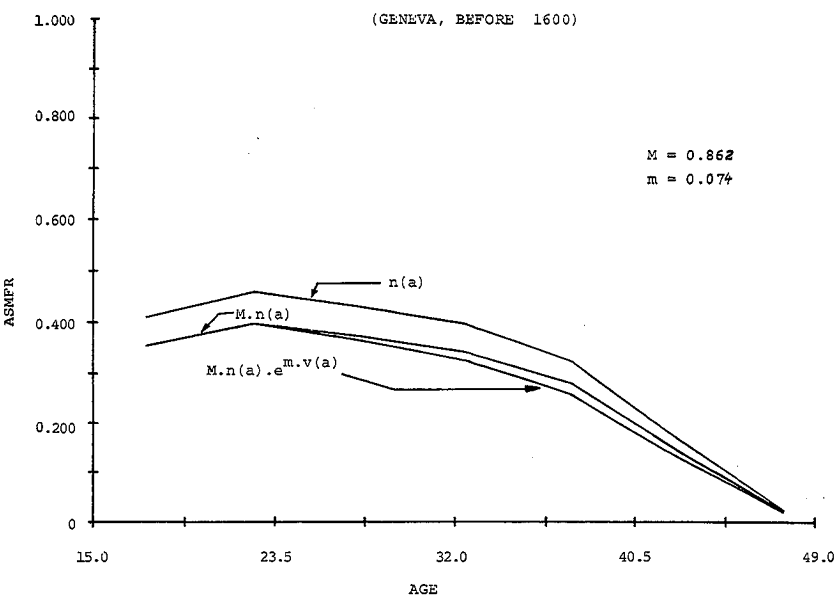

FIGURE 1. COALE-TRUSSELL FERTILITY MODEL

Terminology in this area is not totally clear or consistent, with ambiguous use of such terms as level and pattern. With regard to the age-curve of fertility, however, it is clear that three separate terms are necessary. We suggest speaking of (1) the magnitude or level of fertility (as captured by the total fertility rate or the norm of the rate vector), which in turn reflects (2) scale and (3) pattern or shape. Roy and Nair (1983) speak of level as reflecting volume and pattern, which makes the necessary distinctions. But the term volume is not in common use, and may have unfortunate connotations in this context; for example, it may suggest absolute number, as in migration analysis, or a sum or integral, rather than a constant.

Standardized measures, such as Coale's I_g, reflect the shape of the age curve of fertility in yet a different sense, in that their combination with a population age structure (as the dot product, say, of the two vectors) results in interaction (as measured by Cos θ). More concretely, the age curve of fertility considered by itself has both scale and shape; its shape has further consequences in measures such as I_g. But the inherent shape and the further consequences are separate dimensions and can be measured separately. Separate measures are useful since these further consequences relate to noise or error.

When two demographic measures are compared across populations or across time, their differences or ratio will reflect still further consequences of shape or pattern (for example, cross terms) that in turn are distinct from either inherent shape or interaction effects contained in a single measure.

Some Applications

To return to standardized rates, the interaction term in Equation 4 reminds us that, except for very special substantive or expositional purposes, traditional directly standardized rates may not be either particularly useful nor necessary. If one already knows the specific rates needed for Equation 4, then simpler age-controlled measures such as the unweighted sum, the average or the norm of the rate vector, can be used instead.

The purpose of indirect standardization is different; we resort to it precisely because the actual specific rates are unknown. Thus, it would be useful to have a standardized measure that does not reflect the arbitrary interaction introduced by the choice of standard rates. Equation 5, when rewritten, suggests an approach:

$$|m| = I\ |m^*|\ \frac{\cos \theta^*}{\cos \theta} \tag{6}$$

That is, the norm of the actual rate vector equals the ratio of actual to expected events (I) times the norm of the standard rate vector times the ratio of the cosines of the angles between the actual population vector and the standard and actual rate vectors respectively. On the right of Equation 6, all the terms are known except cos θ. But if it can be guessed or estimated, Equation 6 yields an estimate of $|m|$, an age-standardized summary measure of the unknown actual rate vector. Since cos θ has been little used for these purposes, at the moment we can offer no specific guidelines for its estimation. Empirical work is needed to establish typical values for different demographic phenomena in a wide range of demographic conditions.

Similarly, for direct standardization, a standardized rate adjusted for or purged of the arbitrary interaction resulting from choice of standard would equal (see Equation 4):

$$\frac{m \cdot f^*}{p^*} \times \frac{1}{\cos \varphi} = \frac{|m|\ |f^*|}{p^*}$$

where the first term on the left is the conventional standardized rate and the second a correction term. The term on the right is more convenient for computing the corrected rate. If one is interested in comparing only two populations, another approach would be to select as the "standard population" a vector that makes equal angles with the two rate vectors, given by the formula $w = |v|\ u + |u|\ v$, where w is the new standard vector, and u and v the rate vectors. The cos φ term in Equation 4 will then be equal for the two rates, yielding an unbiased comparison. This approach is not applicable when one wishes to compare several standardized rates.

The usefulness of this system for descriptive purposes can be illustrated further (see our Table 1) with data from Table 1 in Roy and Nair (1983). In this hypothetical example, known age-specific marital fertility rates are assumed equal for two provinces, along with the actual distribution of married women by age relative to the total population. Let f be their "married women per population" and m their "ASMFR" (age-specific marital fertility rates). In vector terms, we get the following results (Table 1).

The summary measure of age-specific marital fertility $|m|$ is the same in both populations, as it should be. The summary measure of relative number of married women $|f|$ is about 29 per cent higher in population A (.0815/.0632; compare this to the sums of their "married women per population": .2030/.1580 = 1.28). The interaction between age patterns of fertility and of married population, as measured by cos θ, is about 12 per cent higher in population A (.9736/.8725). These two factors combine to yield "total births" in A about 44 per cent higher than in B (.0410/.0285). Controlling for relative size of the population of married women by dividing by the sum of "married women per population" yields a "birth rate" about 12 per cent higher in population

TABLE 1. SOME VECTOR MEASURES OF MARITAL FERTILITY

Measure	Population A	Population B	Ratio A/B
$\lvert \underline{m} \rvert$	.5173	.5173	1.00
$\lvert \underline{f} \rvert$	.0815	.0632	1.29
Cos Θ	.9736	.8725	1.12
$\underline{m} \cdot \underline{f}$ (total births)	.0410	.0285	1.44
$\dfrac{\underline{m} \cdot \underline{f}}{\Sigma f_i}$ (birth rate)	.2020	.1804	1.12
Cos Θ^*	.9680	.9703	1.00
Cos Θ/Cos Θ^* (error term)	1.0058	.8992	1.12
Ig	.5048	.4514	1.12
r	.9868	.4262	2.32
r*	.7898	.8079	0.98

Note:
Using data from Roy and Nair (1983), Table 1, p.17. Cos Θ refers to interaction between actual rates and structure, Cos Θ^*, between Hutterite standard rates and actual structure.

A than B (.2020/.1804 = 1.120). In other words, the effects of the interaction of age patterns still remain.

Similarly, Coale's I_g for population A is 12 per cent higher than for population B, even though marital age-specific fertility is identical by assumption and even though I_g is indirectly standardized for age, a result that reflects different amounts of interaction between actual rates and structure and between standard (Hutterite) rates and structure in the two populations. More specifically, the ratio of interaction terms for population A, $\text{Cos}\theta_a/\text{Cos}\theta^*_a$, is 1.0058, implying a small error in I_g. For population B, by contrast, the error term, $\text{Cos}\theta_b/\text{Cos}\theta^*_b$, is 0.8992, and the value of I_g is correspondingly low.

Another way of stating the problem is to note that correlation coefficients (*r*) between population structure and Hutterite rates are similar for populations A and B, whereas the coefficient (*r*) between structure and actual marital fertility is much higher in A than in B.

Coale's I_g

Coale and Trussell have shown (Coale, 1977; Coale and Trussell, 1978) that marital age-specific fertility rates $r(a)$ can be expressed by the following equation:

$$r(a) = M\, n(a) e^{mv(a)} \tag{7}$$

where $n(a)$ is a schedule of natural marital fertility, M is a scale factor expressing the level of marital fertility at about age 20 relative to natural fertility, $v(a)$ gives the typical pattern of reduction of fertility at age a below the natural level, and m is the extent to which control affects the pattern of marital fertility. $v(a)$ and $n(a)$ are derived empirically from sets of natural fertility schedules due to Henry (Coale, 1977:147). M and m are specific to a given set of observed marital fertility rates. In terms of our previous comments, M relates to a dilation or contraction of the natural fertility curve; $e^{mv(a)}$ relates to its "bending."

The Coale-Trussell equation enables us to see how the scale and shape of the fertility curve are implicitly contained in summary measures of the level of fertility, such as the total fertility rate or the norm of the fertility vector. For example, changing their notation to discrete form so that $m_i = Mn_i e^{\beta V_i}$, we can express the total fertility rate as:

$$
\begin{aligned}
TFR &= 5\ \Sigma\ m_i \\
&= 5\ \Sigma\ Mn_ie^{\beta v_i} \\
&= 5\ M\ \Sigma\ n_ie^{\beta v_i}
\end{aligned} \tag{8}
$$

where M reflects scale and $\Sigma\ n_ie^{\beta V_i}$ reflects shape or pattern, the area under the curve (polygon) that would result from only pattern-related departures from the reference curve n_i. Separate measures of these two dimensions, of course, cannot be recovered from the marital total fertility rate alone; we must know the complete schedule, or at least the parameters M and m for the actual fertility schedule in question.

The Coale-Trussell equation and the vector algebra discussed earlier can also be used to decompose Coale's I_g. In his original formulation:

$$I_g = \frac{\Sigma\ m_if_i}{\Sigma\ m^*{}_if_i}$$

where m_i and $m^*{}_i$ are actual and standard (Hutterite) marital fertility rates, and f_i are the number of married females. From Coale-Trussell (with changed notation),

let $m_i = Mn_ie^{\beta V_i} = M\alpha_i$

where n_i corresponds to their $n(a)$, β to their m and α_i is substituted for $n_ie^{\beta V_i}$ for convenience. Then

$$I_g = M\frac{\Sigma\ \alpha_if_i}{\Sigma\ m_if_i} = M\frac{\alpha \cdot f}{m^* \cdot f} \tag{9}$$

$$= M \times \frac{|\alpha|}{|m^*|} \times \frac{Cos\ \theta}{Cos\ \theta^*}\ or\ (M)\ (k\ |\alpha|)\ \frac{(Cos\ \theta)}{Cos\ \theta^*}$$

where $k = 1/\ |m^*|$. Thus, I_g reflects the scale of actual marital fertility as captured by M, the pattern or shape of the actual fertility curve as captured by the ratio of $|\alpha|$ to $|m^*|$ (or $k\ |\alpha|$, since $|m^*|$ is constant), and the ratio of interactions between actual population structure and actual (Cos θ) and standard (Cos θ^*) fertility respectively, an error term.

How serious is this error term? There is no general answer. It depends on the relationship between the shape of the standard Hutterite fertility curve and the actual fertility curve in the population under study. Where the shapes of these curves do not differ radically, the term Cos θ/Cos θ^* will be close to 1.0, and I_g will contain a small error. The formal condition of this result, stated long ago by Yule (1934), is that $m_i/m^*_i = k$ or, in vector terms that m be a dilation or contraction of m^*.

In many concrete situations, this condition will be approached sufficiently that the error in I_g can be ignored. For the Indian states studied by Roy and Nair (1983), for example, the correlation between I_g and the total marital fertility rate in 1961 is +0.992. In other words, the use of an indirectly standardized index in this situation would yield the same substantive results overall as measures based on the actual age-specific rates. On the other hand, in the specific comparison in Roy's and Nair's Table 1, I_g showed marital fertility in population A 12 per cent higher than in population B, even though it was assumed to be identical. Such an error could be substantively misleading.

Particular caution may be warranted in situations where there is reason to suspect unstable observations due to small numbers or highly atypical fertility behaviour. In the case of I_g, for example, extra caution may be needed in studying cities, or rural versus urban populations in the presence of extensive rural-urban migration. Resulting extreme and irregular age, sex and marital structures could so influence marital fertility as to yield a large error term Cos θ/Cos θ^*. This point is at variance with the common advice to use indirect standardization where numbers are small, even if the specific rates are known (for example, see Wunsch and Termote, p. 58). Clearly, some reconciliation is needed.

In the above discussion, we have spoken of the interaction term Cos θ/Cos θ^* as "error" resulting from the "arbitrary" choice of standard. In the case of I_g, the choice of standard (Hutterite) rates was not "arbitrary" in the dictionary sense of "determined by whim or caprice," since Coale purposely chose empirical maximum rates, an appealing reference point. But it remains arbitrary in the lesser sense that use of another set with the same level but different shape would yield Ig's with different error terms due to different amounts of interaction. This is the cost of using quasi-absolute reference points rather than the only truly absolute reference point, namely zero.

Since the Coale-Trussell formula also involves an empirical reference vector of rates $n(a)$, its use also yields numerical results relative to the particular reference vector used. In Equation 9, for example, M, $k|\alpha|$, and Cos θ/Cos θ^* are all functions of n (or $n(a)$ in their notation). Use of a different set of natural fertility rates would yield different numerical results.

Summary and Conclusion

This note suggests that certain problems in traditional standardization can be clarified by treating demographic rates and population structures as vectors, and by making use of the concept of the Euclidean norm or magnitude of a vector as a summary measure of rates, and of the general relationship (where a and b are vectors)

$$a \cdot b = \Sigma\, a_i b_i = |a|\ |b|\ Cos\ \theta$$

where $|a|$ and $|b|$ are the magnitudes of the vectors and θ is the angle between them. In the context of standardization, Cos θ provides a convenient summary measure of error associated with the arbitrary choice of standard (rates or structure), error due to interaction between the actual and standard vectors.

Use of this relationship and of the Coale-Trussell model of marital fertility yields a decomposition of Coale's I_g into terms reflecting scale, pattern and interaction or error.

This analysis is a reminder that any system of measurement or modelling that uses a reference point other than zero yields numerical results relative to the reference point chosen. A practical implication is that, to ensure comparability of results, investigators should adhere to established reference points (for example, Hutterite fertility or Coale and Trussell's average natural fertility rates), unless they have compelling reasons for not doing so.

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